

Introduction to Algebraic Topology

MAST31023

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CONTENTS

0. INTRODUCTION

These notes cover a one-semester basic course in algebraic topology. The course begins by introducing some fundamental notions as categories, functors, homotopy, contractibility, paths, path components and simplexes. After that we will study the fundamental group; the Fundamental Theorem of Algebra will be proved as an application. This will take roughly the first half of the semester. During the second half of the semester we will study singular homology. We will show that singular homology satisfies the Eilenberg-Steenrod Axioms. As an application we will, for example, prove the Brouwer Fixed Point Theorem in all dimensions. We will finish the semester by doing more applications of singular homology, exactly what topics will be covered depends on how much time will be left.

These notes are based on Joseph Rotman's book "An Introduction to Algebraic Topology".

1. CATEGORIES AND FUNCTORS

Definition 1.1. A *category* $\mathcal{C}$ consists of three ingredients:

- (1) a class of *objects*, $\text{obj}(\mathcal{C})$,
- (2) a class of *morphisms*, $\text{Hom}(\mathcal{C})$,
- (3) *composition* of morphisms.

For every ordered pair (A, B) of objects in $\mathcal{C}$, there is a *set* of morphisms $\text{Hom}(A, B)$.

For objects A, B, C of $\mathcal{C}$, there is composition of morphisms

$$\text{Hom}(A, B) \times \text{Hom}(B, C) \rightarrow \text{Hom}(A, C), \quad (f, g) \mapsto g \circ f.$$

The following axioms are satisfied:

- (1) the sets $\text{Hom}(A, B)$ are pairwise disjoint,
- (2) composition is associative

$$(h \circ (g \circ f)) = (h \circ g) \circ f,$$

- (3) for every object A of $\mathcal{C}$, there exists a morphism $1_A: A \rightarrow A$ called the *identity morphism* for A , such that $1_A \circ f = f$, for every $f \in \text{Hom}(B, A)$ and for every object B of $\mathcal{C}$, and $g \circ 1_A = g$, for every $g \in \text{Hom}(A, C)$ and for every object C of $\mathcal{C}$.

We often also denote the identity morphism 1_A by id_A .

Example 1.2. The *category of all sets* is $\mathcal{C} = \text{Sets}$:

- (1) $\text{obj}(\mathcal{C}) = \text{all sets}$,
- (2) $\text{Hom}(A, B) = \text{the family of all functions } A \rightarrow B$,
- (3) composition is the usual composition of functions.

Example 1.3. The *category of all topological spaces* is $\mathcal{C} = \text{Top}$:

- (1) $\text{obj}(\mathcal{C}) = \text{all topological spaces}$,
- (2) $\text{Hom}(A, B) = \text{the family of all continuous functions } A \rightarrow B$,
- (3) composition is the usual composition of functions.

Although the objects of a category $\mathcal{C}$ do not necessarily form a set, we use the expression $A \in \text{obj}(\mathcal{C})$ to mean that A is an object of $\mathcal{C}$. Similarly, if $\mathcal{A}$ is another category, we write $\text{obj}(\mathcal{C}) \subset \text{obj}(\mathcal{A})$ to mean that every object in $\mathcal{C}$ is also an object in $\mathcal{A}$.

Definition 1.4. Let $\mathcal{A}$ and $\mathcal{C}$ be categories. Assume $\text{obj}(\mathcal{C}) \subset \text{obj}(\mathcal{A})$. For $A, B \in \text{obj}(\mathcal{C})$, we denote the sets of morphisms corresponding to $\mathcal{A}$ and $\mathcal{C}$ by

$$\text{Hom}_{\mathcal{A}}(A, B) \text{ and } \text{Hom}_{\mathcal{C}}(A, B),$$

respectively. We call $\mathcal{C}$ a *subcategory* of $\mathcal{A}$, if

$$\text{Hom}_{\mathcal{C}}(A, B) \subset \text{Hom}_{\mathcal{A}}(A, B),$$

for all $A, B \in \text{obj}(\mathcal{C})$ and if the composition in $\mathcal{C}$ is the same as the composition in $\mathcal{A}$.

Example 1.5. Subcategories of Top: We obtain subcategories by restriction. For example, the objects could be all Hausdorff spaces, all compact spaces or all connected spaces. If we choose the objects to be smooth manifolds, it makes sense to choose the morphisms to be smooth maps.

Example 1.6. The *category of all groups* is $\mathcal{C} = \text{Groups}$:

- (1) $\text{obj}(\mathcal{C}) = \text{all groups}$,
- (2) $\text{Hom}(A, B) = \text{the family of all homomorphisms } A \rightarrow B$,
- (3) composition is the usual composition of functions.

Example 1.7. The *category of all abelian groups* is $\mathcal{C} = \text{Ab}$:

- (1) $\text{obj}(\mathcal{C}) = \text{all abelian groups}$,
- (2) $\text{Hom}(A, B) = \text{the family of all homomorphisms } A \rightarrow B$,
- (3) composition is the usual composition of functions.

Then Ab is a subcategory of Groups .

Example 1.8. The *category of all rings* is $\mathcal{C} = \text{Rings}$:

- (1) $\text{obj}(\mathcal{C}) = \text{all rings}$,
- (2) $\text{Hom}(A, B) = \text{the family of all ring homomorphisms } A \rightarrow B$, that preserve identity elements,
- (3) composition is the usual composition of functions.

Example 1.9. The category $\mathcal{C} = \text{Top}^2$:

- (1) $\text{obj}(\mathcal{C}) = \text{all ordered pairs } (X, A)$, where X is a topological space and A is a subspace of X ,
- (2) $\text{Hom}((X, A), (Y, B))$: A morphism $f: (X, A) \rightarrow (Y, B)$ is a continuous map $f: X \rightarrow Y$ such that $f(A) \subset B$,
- (3) composition is the usual composition of functions.

Example 1.10. The *category of pointed spaces* is Top_* . The objects of this category are all ordered pairs (X, x_0) where X is a topological space and $x_0 \in X$. A morphism $f: (X, x_0) \rightarrow (Y, y_0)$ is a continuous map $f: X \rightarrow Y$ such that $f(x_0) = y_0$. We call x_0 the *basepoint* of X . Morphisms of this category are called

pointed maps or *basepoint preserving maps*. Objects are called *pointed spaces*. The category Top_* is a subcategory of Top^2 .

Definition 1.11. Let $\mathcal{C}$ be a category. Let $\sim$ be an equivalence relation on

$$\bigcup_{(A,B)} \text{Hom}(A, B).$$

We call $\sim$ a *congruence* on $\mathcal{C}$, if it satisfies the following conditions:

- (1) if $f \in \text{Hom}(A, B)$ and $f \sim f'$, then $f' \in \text{Hom}(A, B)$,
- (2) if $f \sim f'$, $g \sim g'$ and the composite $g \circ f$ exists, then $g \circ f \sim g' \circ f'$.

The proof of the following theorem follows immediately from the definitions.

Theorem 1.12. Let $\mathcal{C}$ be a category and let $\sim$ be a congruence on $\mathcal{C}$. Let $[f]$ denote the equivalence class of a morphism f . Define $\mathcal{C}'$ by:

- (1) $\text{obj}(\mathcal{C}') = \text{obj}(\mathcal{C})$,
- (2) $\text{Hom}_{\mathcal{C}'}(A, B) = \{[f] \mid f \in \text{Hom}_{\mathcal{C}}(A, B)\}$,
- (3) $[g] \circ [f] = [g \circ f]$.

Then $\mathcal{C}'$ is a category.

The category $\mathcal{C}'$ is called a *quotient category* of $\mathcal{C}$. For us the most important quotient category will be the *homotopy category* defined later.

Definition 1.13. Let $\mathcal{A}$ and $\mathcal{C}$ be categories. Let $T: \mathcal{A} \rightarrow \mathcal{C}$ satisfy the following:

- (1) if $A \in \text{obj}(\mathcal{A})$, then $TA \in \text{obj}(\mathcal{C})$,
- (2) if $f: A \rightarrow A'$ is a morphism in $\mathcal{A}$, then $Tf: TA \rightarrow TA'$ is a morphism in $\mathcal{C}$,
- (3) if f and g are morphisms in $\mathcal{A}$ and $g \circ f$ is defined, then $T(g \circ f) = Tg \circ Tf$,
- (4) $T1_A = 1_{TA}$, for every $A \in \text{obj}(\mathcal{A})$.

We say that T is a (*covariant*) *functor* from $\mathcal{A}$ to $\mathcal{C}$.

Example 1.14. Let $\mathcal{A}$ and $\mathcal{C}$ be categories and let $T: \mathcal{A} \rightarrow \mathcal{C}$ be a functor. We call T a *forgetful functor*, if it "forgets" some of the structure or properties of $\mathcal{A}$. The functor $T: \text{Top} \rightarrow \text{Sets}$ that assigns to each topological space its underlying set and to each continuous function itself is an example of a forgetful functor.

Example 1.15. Let $\mathcal{C}$ be a category. The *identity functor* $J: \mathcal{C} \rightarrow \mathcal{C}$ is defined by $JA = A$ for every $A \in \text{obj}(\mathcal{C})$ and $Jf = f$ for every morphism f .

Example 1.16. Let Y be a topological space. Then there is a functor

$$T_Y: \text{Top} \rightarrow \text{Top},$$

where $T_Y(X) = X \times Y$, for a topological space X , and, for a continuous function $f: X \rightarrow X'$,

$$T_Y(f): X \times Y \rightarrow X' \times Y,$$

is defined by $(x, y) \mapsto (f(x), y)$.

The functor $\text{Hom}(A, _)$ in the following example is called a *covariant Hom functor*.

Example 1.17. Let $\mathcal{C}$ be a category. Let $A \in \text{obj}(\mathcal{C})$. Define a functor

$$\text{Hom}(A, _): \mathcal{C} \rightarrow \text{Sets}$$

as follows: assign the set $\text{Hom}(A, B)$ to each $B \in \text{obj}(\mathcal{C})$, and assign the induced map

$$\text{Hom}(A, f): \text{Hom}(A, B) \rightarrow \text{Hom}(A, B'), \quad g \mapsto f \circ g,$$

to every morphism $f: B \rightarrow B'$.

A functor is called *contravariant*, if it reverses the direction of arrows:

Definition 1.18. Let $\mathcal{A}$ and $\mathcal{C}$ be categories. Let $S: \mathcal{A} \rightarrow \mathcal{C}$ satisfy the following:

- (1) if $A \in \text{obj}(\mathcal{A})$, then $SA \in \text{obj}(\mathcal{C})$,
- (2) if $f: A \rightarrow A'$ is a morphism in $\mathcal{A}$, then $Sf: SA' \rightarrow SA$ is a morphism in $\mathcal{C}$,
- (3) if f and g are morphisms in $\mathcal{A}$ and $g \circ f$ is defined, then $S(g \circ f) = Sf \circ Sg$,
- (4) $S1_A = 1_{SA}$, for every $A \in \text{obj}(\mathcal{A})$.

We say that S is a *contravariant functor* from $\mathcal{A}$ to $\mathcal{C}$.

Example 1.19. Let $\mathcal{C}$ be a category. Let $B \in \text{obj}(\mathcal{C})$. Define a functor

$$\text{Hom}(_, B): \mathcal{C} \rightarrow \text{Sets}$$

as follows: assign the set $\text{Hom}(A, B)$ to each $A \in \text{obj}(\mathcal{C})$, and assign the induced map

$$\text{Hom}(g, B): \text{Hom}(A', B) \rightarrow \text{Hom}(A, B), \quad h \mapsto h \circ g,$$

to every morphism $g: A \rightarrow A'$.

The functor $\text{Hom}(_, B)$ is called a *contravariant Hom functor*.

Definition 1.20. Let $\mathcal{C}$ be a category and let $A, B \in \text{obj}(\mathcal{C})$. Let $f: A \rightarrow B$ be a morphism. If there is a morphism $g: B \rightarrow A$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$, we say that f is an *equivalence* or an *isomorphism* in $\mathcal{C}$.

Theorem 1.21. Let $\mathcal{A}$ and $\mathcal{C}$ be categories, and let $T: \mathcal{A} \rightarrow \mathcal{C}$ be either a covariant or a contravariant functor. Let f be an equivalence in $\mathcal{A}$. Then Tf is an equivalence in $\mathcal{C}$.

Proof. Assume T is covariant. Let $f: A \rightarrow B$ be a morphism that is an equivalence in $\mathcal{A}$. Then there is a morphism $g: B \rightarrow A$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$. Since T is covariant, it follows that

$$Tg \circ Tf = T(g \circ f) = T1_A = 1_{TA},$$

and

$$Tf \circ Tg = T(f \circ g) = T1_B = 1_{TB}.$$

Thus Tf is an equivalence in $\mathcal{C}$. The case where T is contravariant is similar. $\square$

2. HOMOTOPY

Definition 2.1. Let X and Y be topological spaces and let f_0 and f_1 be continuous maps from X to Y . We say that f_0 is *homotopic* to f_1 (denoted by $f_0 \simeq f_1$), if there is a continuous map

$$F: X \times I \rightarrow Y$$

such that

$$F(x, 0) = f_0(x) \text{ and } F(x, 1) = f_1(x) \text{ for all } x \in X.$$

The map F is called a *homotopy*.

If F is a homotopy from f_0 to f_1 , we often write $F: f_0 \simeq f_1$.

Let $t \in I$. Define $f_t: X \rightarrow Y$ by $f_t(x) = F(x, t)$. Then the homotopy F gives a *one-parameter family* of continuous maps deforming f_0 into f_1 .

Recall the following *gluing lemmas*:

Lemma 2.2. (*Gluing lemma 1*) Let X be a topological space and let X_i , $1 \leq i \leq n$, be finitely many closed subsets of X such that $X = \bigcup_{i=1}^n X_i$. Let Y be a topological space. Assume there are continuous maps $f_i: X_i \rightarrow Y$, $1 \leq i \leq n$, such that

$$f_i|_{X_i \cap X_j} = f_j|_{X_i \cap X_j}, \text{ for all } i, j.$$

Then there exists a unique continuous map $f: X \rightarrow Y$ with $f|_{X_i} = f_i$ for all i .

Lemma 2.3. (*Gluing lemma 2*) Let X be a topological space and let X_i be (possibly infinitely many) open subsets of X such that $X = \bigcup_i X_i$. Let Y be a topological space. Assume there are continuous maps $f_i: X_i \rightarrow Y$ such that

$$f_i|_{X_i \cap X_j} = f_j|_{X_i \cap X_j}, \text{ for all } i, j.$$

Then there exists a unique continuous map $f: X \rightarrow Y$ with $f|_{X_i} = f_i$ for all i .

The proofs of Lemmas 2.2 and 2.3 are easy and can be found in many general topology text books.

Theorem 2.4. *Homotopy is an equivalence relation on the set of all continuous maps $X \rightarrow Y$.*

Proof. Let $f: X \rightarrow Y$ be a continuous map. Then

$$F: X \times I \rightarrow Y, (x, t) \mapsto f(x),$$

for all $x \in X$ and for all $t \in I$, is a homotopy, $F: f \simeq f$. Thus $\simeq$ is reflexive.

Let next $f, g: X \rightarrow Y$ and assume $f \simeq g$. Then there is a continuous map $F: X \times I \rightarrow Y$, such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$, for all $x \in X$. Let

$$G: X \times I \rightarrow Y, (x, t) \mapsto F(x, 1 - t).$$

Then $G: g \simeq f$. It follows that $\simeq$ is symmetric.

Finally, let $f, g, h: X \rightarrow Y$ and assume $F: f \simeq g$ and $G: g \simeq h$. Let

$$H: X \times I \rightarrow Y, (x, t) \mapsto \begin{cases} F(x, 2t), & \text{if } 0 \leq t \leq \frac{1}{2}, \\ G(x, 2t - 1), & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Now, for $t = \frac{1}{2}$,

$$F(x, 2 \cdot \frac{1}{2}) = F(x, 1) = g(x), \text{ for every } x \in X,$$

and

$$G(x, 2 \cdot \frac{1}{2} - 1) = G(x, 0) = g(x), \text{ for every } x \in X.$$

Thus it follows from the first gluing lemma that H is continuous. Therefore, $H: f \simeq h$. Consequently, the relation $\simeq$ is transitive. $\square$

Definition 2.5. The *homotopy class* of a continuous map $f: X \rightarrow Y$ is the equivalence class

$$[f] = \{\text{continuous } g: X \rightarrow Y \mid g \simeq f\}.$$

The family of all such homotopy classes is denoted by $[X, Y]$.

Theorem 2.6. Let $f_0, f_1: X \rightarrow Y$ and $g_0, g_1: Y \rightarrow Z$ be continuous. Assume $f_0 \simeq f_1$ and $g_0 \simeq g_1$. Then $g_0 \circ f_0 \simeq g_1 \circ f_1$, i.e., $[g_0 \circ f_0] = [g_1 \circ f_1]$.

Proof. Let $F: f_0 \simeq f_1$ and $G: g_0 \simeq g_1$ be homotopies. Let

$$H: X \times I \rightarrow Z, (x, t) \mapsto G(f_0(x), t).$$

Then H is continuous,

$$H(x, 0) = G(f_0(x), 0) = g_0(f_0(x)),$$

and

$$H(x, 1) = G(f_0(x), 1) = g_1(f_0(x)),$$

for all $x \in X$. Thus $H: g_0 \circ f_0 \simeq g_1 \circ f_0$. Let then

$$K: X \times I \rightarrow Z, (x, t) \mapsto (g_1 \circ F)(x, t).$$

Thus $K: g_1 \circ f_0 \simeq g_1 \circ f_1$. Since homotopy is a transitive relation, it follows that $g_0 \circ f_0 \simeq g_1 \circ f_1$. $\square$

The following corollary follows immediately from the definition of congruence and from Theorems 2.4 and 2.6:

Corollary 2.7. *Homotopy is a congruence on the category Top.*

According to Theorem 1.12, there is a quotient category whose objects are topological spaces and whose morphism sets are $\text{Hom}(X, Y) = [X, Y]$. The composition is given by $[g] \circ [f] = [g \circ f]$. This category is called the *homotopy category* and it is denoted by hTop .

Definition 2.8. A continuous map $f: X \rightarrow Y$ is called a *homotopy equivalence*, if there exists a continuous map $g: Y \rightarrow X$ with $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$. Topological spaces X and Y have the same *homotopy type*, if there is a homotopy equivalence $f: X \rightarrow Y$.

Clearly, $f: X \rightarrow Y$ is a homotopy equivalence, if and only if $[f] \in [X, Y]$ is an equivalence in hTop .

Definition 2.9. Let X and Y be topological spaces and let $y_0 \in Y$. The map $c: X \rightarrow Y$, $x \mapsto y_0$, is called the *constant map at y_0* . A continuous map $f: X \rightarrow Y$ is called *nullhomotopic*, if there exists a constant map $c: X \rightarrow Y$ with $f \simeq c$.

Theorem 2.10. Let Y be a topological space and let $f: \mathbb{S}^n \rightarrow Y$ be a continuous map. The following conditions are equivalent:

- (1) the map f is nullhomotopic,
- (2) the map f can be extended to a continuous map $\mathbb{D}^{n+1} \rightarrow Y$,
- (3) if $x_0 \in \mathbb{S}^n$ and $k: \mathbb{S}^n \rightarrow Y$ is the constant map at $f(x_0)$, then there is a homotopy $F: f \simeq k$ with $F(x_0, t) = f(x_0)$ for all $t \in I$.

Proof. We first show that (1) implies (2): Let $c: \mathbb{S}^n \rightarrow Y$ be the constant map at $y_0 \in Y$. Assume $F: f \simeq c$. Let

$$g: \mathbb{D}^{n+1} \rightarrow Y, \quad x \mapsto \begin{cases} y_0, & \text{if } 0 \leq \|x\| \leq \frac{1}{2}, \\ F(\frac{x}{\|x\|}, 2 - 2\|x\|), & \text{if } \frac{1}{2} \leq \|x\| \leq 1. \end{cases}$$

The map g is well defined, since for $\|x\| = \frac{1}{2}$,

$$F(\frac{x}{\|x\|}, 2 - 2\|x\|) = F(2x, 1) = y_0.$$

By the first gluing lemma, g is continuous. If $x \in \mathbb{S}^n$, then $\|x\| = 1$ and

$$g(x) = F(x, 2 - 2 \cdot 1) = F(x, 0) = f(x).$$

Thus g is an extension of f .

We next show that (2) implies (3): Assume that $g: \mathbb{D}^{n+1} \rightarrow Y$ extends f . Let

$$F: \mathbb{S}^n \times I \rightarrow Y, \quad (x, t) \mapsto g((1 - t)x + tx_0).$$

Clearly, F is continuous. For all $x \in \mathbb{S}^n$,

$$F(x, 0) = g(x) = f(x) \quad \text{and} \quad F(x, 1) = g(x_0) = f(x_0).$$

Thus $F: f \simeq k$, where $k: \mathbb{S}^n \rightarrow Y$, $x \mapsto f(x_0)$. For all $t \in I$,

$$F(x_0, t) = g((1 - t)x_0 + tx_0) = g(x_0) = f(x_0).$$

Obviously, (3) implies (1). □

3. CONVEXITY, CONTRACTIBILITY AND CONES

Definition 3.1. A subset X of $\mathbb{R}^m$ is called *convex*, if $tx + (1 - t)y \in X$, for all $x, y \in X$ and for all $t \in I$.

Definition 3.2. A topological space X is called *contractible*, if the identity map $1_X: X \rightarrow X$ is nullhomotopic.

Theorem 3.3. Every convex set is contractible.

Proof. Let X be a convex set and let $x_0 \in X$. Define $c: X \rightarrow X$ by $c(x) = x_0$ for all $x \in X$. Define $F: X \times I \rightarrow X$ by $F(x, t) = tx_0 + (1 - t)x$. Then $F: 1_X \simeq c$. □

Definition 3.4. Let X be a topological space and let $X' = \{X_j \mid j \in J\}$ be a collection of subsets of X . We call X' a *partition* of X , if the following conditions hold:

- (1) $X_j \neq \emptyset$, for every $j \in J$,
- (2) $X = \bigcup_{j \in J} X_j$,
- (3) $X_i \cap X_j = \emptyset$, for all $i, j \in J$, where $i \neq j$.

Definition 3.5. Let X be a topological space and let $X' = \{X_j \mid j \in J\}$ be a partition of X . The map

$$\nu: X \rightarrow X', \quad x \mapsto X_j, \quad \text{if } x \in X_j,$$

is called the *natural map* (or *natural projection* or *quotient map*). The *quotient topology* on X' is the collection of all subsets U' of X' such that $\nu^{-1}(U')$ is open in X .

Notice that the natural map ν is always a continuous surjection.

Example 3.6. Let X and Y be topological spaces, and let $f: X \rightarrow Y$ be a function. Define $x \sim x'$ if $f(x) = f(x')$. Then $\sim$ is an equivalence relation. Let $\nu: X \rightarrow X/\ker f$ be the natural map, where $X/\ker f$ denotes the quotient space. Let $[x]$ denote the equivalence class of $x \in X$. The map

$$\varphi: X/\ker f \rightarrow Y, \quad [x] \mapsto f(x),$$

is an injection making the diagram

$$\begin{array}{ccc} X & & \\ \nu \downarrow & \searrow f & \\ X/\ker f & \xrightarrow{\varphi} & Y \end{array}$$

commute.

Definition 3.7. A continuous surjection $f: X \rightarrow Y$ is called an *identification* if it satisfies the following: A subset U of Y is open if and only if $f^{-1}(U)$ is open in X .

Example 3.8. Let $\sim$ be an equivalence relation on a topological space X and let $X/\sim$ be equipped with the quotient topology. Then the natural map $\nu: X \rightarrow X/\sim$ is an identification.

Example 3.9. Let $f: X \rightarrow Y$ be a continuous map that is a surjection. If f is either open or closed, then it is an identification.

Let $f: X \rightarrow Y$ be a continuous map. Assume there is a continuous map $s: Y \rightarrow X$ with $f \circ s = 1_Y$. We call s a *section* of f . Notice that f must be surjective in order to have a section.

Example 3.10. A continuous map $f: X \rightarrow Y$ having a section is an identification.

Theorem 3.11. Let $f: X \rightarrow Y$ be a continuous surjection. Then f is an identification if and only if the following holds: For all spaces Z and for all functions $g: Y \rightarrow Z$, the function g is continuous if and only if $g \circ f$ is continuous.

Proof. Let $g: Y \rightarrow Z$ be a function. Then the diagram

$$\begin{array}{ccc} X & & \\ f \downarrow & \searrow g \circ f & \\ Y & \xrightarrow{g} & Z \end{array}$$

commutes. Assume first that f is an identification. Then $g \circ f$ is continuous, if g is continuous. Assume $g \circ f$ is continuous. Let V be an open set in Z . Then

$$(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$$

is open in X . Since f is an identification, it follows that $g^{-1}(V)$ is open in Y . Since V was chosen arbitrarily, it follows that g is continuous.

Assume then that the condition holds. Let $Z = X/\ker f$ and let $\nu: X \rightarrow X/\ker f$ be the natural map. Let $\varphi: X/\ker f \rightarrow Y$ be the injection $[x] \mapsto f(x)$. Since f is a surjection, also φ is a surjection. The diagram

$$\begin{array}{ccc} X & & \\ \nu \downarrow & \searrow f & \\ X/\ker f & \xrightarrow{\varphi} & Y \end{array}$$

commutes. Now, $\nu = \varphi^{-1} \circ f$ is continuous. Since the condition holds, it follows that the inverse function $\varphi^{-1}: Y \rightarrow X/\ker f$ of φ is continuous. Since ν is an identification, it follows that φ is continuous. Thus φ is a homeomorphism and, consequently, f is an identification. $\square$

Definition 3.12. Let $f: X \rightarrow Y$ be a function and let $y \in Y$. Then $f^{-1}(y)$ is called the *fiber* over y .

Corollary 3.13. Let X, Y and Z be topological spaces and let $f: X \rightarrow Y$ be an identification. Let $h: X \rightarrow Z$ be a continuous function that is constant on each fiber of f . Then $g: Y \rightarrow Z, y \mapsto h(f^{-1}(y))$, is continuous. Moreover, the following are equivalent:

- (1) the map g is open (closed),
- (2) the set $h(U)$ is open (closed) in Z whenever U is an open (closed) set in X with $U = f^{-1}(f(U))$.

Proof. The following diagram commutes:

$$\begin{array}{ccc} X & & \\ f \downarrow & \searrow h & \\ Y & \xrightarrow{g} & Z \end{array}$$

Notice that the function g is well defined, since h is constant on each fiber of f . The function $g \circ f = h$ is continuous. Therefore, by Theorem 3.11, also g is continuous.

Assume then that g is an open map. Let U be an open subset of X with $U = f^{-1}(f(U))$. Since f is an identification, it follows that $f(U)$ is open in Y . Since g is open, it follows that $g(f(U))$ is open in Z . But then, $h(U) = g(f(U))$ is open in Z . Thus the first condition implies the second one.

Finally, assume that the second condition holds. Let V be an open subset of Y . Since f is a surjection, $f(f^{-1}(V)) = V$. Thus $f^{-1}(f(f^{-1}(V))) = f^{-1}(V)$. By the second condition, $g(V) = h(f^{-1}(V))$ is open. Thus g is an open map. Therefore, the second condition implies the first one. $\square$

Corollary 3.14. *Let X and Z be topological spaces and let $h: X \rightarrow Z$ be an identification. Then the map*

$$\varphi: X/\ker h \rightarrow Z, \quad [x] \mapsto h(x),$$

is a homeomorphism.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} X & & \\ \nu \downarrow & \searrow h & \\ X/\ker h & \xrightarrow{\varphi} & Z \end{array}$$

By an earlier example, φ is an injection. Since h is a surjection, also φ is a surjection. Since ν is an identification and h is continuous, it follows that φ is continuous. To show that φ is a homeomorphism, it suffices to show that it is an open map. Let U be an open subset of $X/\ker h$. Then

$$h^{-1}(\varphi(U)) = (\varphi \circ \nu)^{-1}(\varphi(U)) = \nu^{-1}(\varphi^{-1}(\varphi(U))) = \nu^{-1}(U)$$

is open in X , since ν is continuous. Since h is an identification, it follows that $\varphi(U)$ is open in Z . Thus φ is a homeomorphism. $\square$

Remark 3.15. Let $f: X \rightarrow W$ and $g: Y \rightarrow Z$ be identifications. Then the cartesian product

$$f \times g: X \times Y \rightarrow W \times Z, \quad (x, y) \mapsto (f(x), g(y)),$$

does not need to be an identification, see [2], Chapter 2, Example 22.7.

However, the following result holds:

Theorem 3.16. *Let X , X' and Z be topological spaces. Assume that Z is locally compact Hausdorff. Let $p: X \rightarrow X'$ be an identification. Then also*

$$p \times \text{id}_Z: X \times Z \rightarrow X' \times Z$$

is an identification.

Proof. Since p and id_Z are continuous, also $p \times \text{id}_Z$ is continuous. Thus $(p \times \text{id}_Z)^{-1}(U')$ is open in $X \times Z$ for every open subset U' of $X' \times Z$.

Assume then that $U' \subset X' \times Z$ and that $U = (p \times \text{id}_Z)^{-1}(U')$ is open. It suffices to show that U' is open. Let $(x', z') \in U'$, and let $x \in X$ be such that $p(x) = x'$. Then $(p \times \text{id}_Z)(x, z) = (x', z)$, which implies that $(x, z) \in U$. Since U is open in

$X \times Z$, it follows that x has an open neighborhood V in X and z has an open neighborhood J in Z with $(x, z) \in V \times J \subset U$. Since Z is locally compact, there is an open subset W of Z such that $z \in W \subset \overline{W} \subset J$ and $\overline{W}$ is compact. Then $\{x\} \times \overline{W} \subset V \times J \subset U$. Let

$$A = \{\alpha \in X \mid \{\alpha\} \times \overline{W} \subset U\}.$$

Then $x \in A$. We show that A is open in X . Let $\alpha \in A$. For every $\xi \in \overline{W}$, there are open subsets L_ξ of X and N_ξ of Z with $(\alpha, \xi) \in L_\xi \times N_\xi \subset U$. Since $\overline{W}$ is compact, the open cover $\{N_\xi \mid \xi \in \overline{W}\}$ has a finite subcover $\{N_1, \dots, N_m\}$. Then, for every $i \in \{1, \dots, m\}$, $L_i \times N_i \subset U$, where we define $L_i = L_\xi$, if $N_i = N_\xi$. Also, $\alpha \in \bigcap_{i=1}^m L_i$ and $\overline{W} \subset \bigcup_{i=1}^m N_i$. Thus $\bigcap_{i=1}^m L_i = L$ is open and $\alpha \in L$, $L \times N_i \subset U$, for every i . Hence

$$L \times \overline{W} \subset \bigcup_{i=1}^m (L \times N_i) \subset U.$$

Consequently, $\alpha \in L \subset A$. It follows that A is open in X .

For $\beta \in X$, $\{\beta\} \times \overline{W} \subset U = (p \times \text{id}_Z)^{-1}(U')$ if and only if $\{p(\beta)\} \times \overline{W} \subset U'$. In particular, $\beta \in A$ if and only if $\{p(\beta)\} \times \overline{W} \subset U'$. Clearly, $A \subset p^{-1}(p(A))$. Assume $\beta \in p^{-1}(p(A))$. Then $p(\beta) \in p(A)$. Thus $p(\beta) = p(\alpha)$, for some $\alpha \in A$. Hence $\{p(\beta)\} \times \overline{W} = \{p(\alpha)\} \times \overline{W} \subset U'$. But this now implies that $\beta \in A$. Therefore, $p^{-1}(p(A)) \subset A$ and, consequently, $A = p^{-1}(p(A))$. Since p is an identification and A is open, it follows that $p(A)$ is open. Thus $p(A) \times W \subset U'$ is an open neighborhood of (x', z) . Hence U' is open. $\square$

The proof of the following useful lemma is similar to that of Theorem 3.16:

Lemma 3.17. (*Tube lemma*) Let X and Y be topological spaces. Assume Y is compact. Let $x_0 \in X$ and let U be an open subset of $X \times Y$ such that $\{x_0\} \times Y \subset U$. Then x_0 has an open neighborhood L in X with $\{x_0\} \times Y \subset L \times Y \subset U$.

Proof. [2], Lemma 3.26.8. $\square$

Corollary 3.18. Let $f: X \rightarrow Y$ be an identification. Then

$$f \times \text{id}_I: X \times I \rightarrow Y \times I$$

is an identification.

Definition 3.19. Let X be a topological space. The relation $(x, t) \sim (x', t')$, for $t = t' = 1$, (and $(x, t) \sim (x, t)$) is an equivalence relation on $X \times I$. The quotient space $X \times I / \sim$ is called the *cone* over X and denoted by CX .

Example 3.20. Let X and Y be topological spaces and let $y_0 \in Y$. Let $f: X \times I \rightarrow Y$ satisfy $f(x, 1) = y_0$ for all $x \in X$. Then f induces a continuous map

$$\bar{f}: CX \rightarrow Y, [x, t] \mapsto f(x, t).$$

Choose $X = \mathbb{S}^n$, $Y = \mathbb{D}^{n+1}$ and

$$f: \mathbb{S}^n \times I \rightarrow \mathbb{D}^{n+1}, (u, t) \mapsto (1 - t)u.$$

Then $f(u, 1) = 0$ for all $u \in \mathbb{S}^n$ and there is a continuous map

$$\bar{f}: C\mathbb{S}^n \rightarrow \mathbb{D}^{n+1}, [u, t] \mapsto (1 - t)u.$$

The map $\bar{f}$ is a homeomorphism. Thus we may consider $\mathbb{D}^{n+1}$ as the cone over $\mathbb{S}^n$ with vertex 0.

Theorem 3.21. *The cone CX is contractible for every topological space X .*

Proof. Let $\nu: X \times I \rightarrow CX$ be the quotient map, and let

$$f: X \times I \times I \rightarrow X \times I, (x, t, s) \mapsto (x, (1 - s)t + s),$$

and

$$F: CX \times I \rightarrow CX, ([x, t], s) \mapsto [x, (1 - s)t + s].$$

Then $\nu \circ f = F \circ (\nu \times 1_I)$. By Corollary 3.18, $\nu \times 1_I$ is an identification. Since ν and f are continuous, it now follows that F is continuous. Since F is continuous, $F_0 = 1_{CX}$ and F_1 equals the constant map taking every point of CX to the vertex point, it follows that CX is contractible. $\square$

Theorem 3.22. *A topological space X is contractible if and only if it has the same homotopy type as a point.*

Proof. Let $\{a\}$ be a one-point space. Assume first that X and $\{a\}$ have the same homotopy type. Then there are maps $f: X \rightarrow \{a\}$ and $g: \{a\} \rightarrow X$ with $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_{\{a\}}$ (in fact, $f \circ g = 1_{\{a\}}$). Now, g is the map $\{a\} \rightarrow X$, $a \mapsto x_0$, for some $x_0 \in X$. Thus $(g \circ f)(x) = g(f(x)) = g(a) = x_0$, for all $x \in X$. Hence $g \circ f$ is a constant map and $1_X \simeq g \circ f$ is nullhomotopic.

Assume then that X is contractible. Then $1_X \simeq k$, where $k: X \rightarrow X$, $x \mapsto x_0$, for some $x_0 \in X$. Let $f: X \rightarrow \{x_0\}$ be the constant map and let $g: \{x_0\} \rightarrow X$, $x_0 \mapsto x_0$. Then $f \circ g = 1_{\{x_0\}}$ and $g \circ f = k \simeq 1_X$. Thus X and $\{x_0\}$ have the same homotopy type. $\square$

4. PATHS AND PATH COMPONENTS

Definition 4.1. A *path* in a topological space X is a continuous map $f: I \rightarrow X$. If $f(0) = a$ and $f(1) = b$, we say that f is a *path from a to b* .

Notice that if f is a path in X from a to b , then $g: I \rightarrow X$, $t \mapsto f(1 - t)$, is a path in X from b to a .

Definition 4.2. A topological space X is *path-connected*, if, for every $a, b \in X$, there is a path from a to b .

Theorem 4.3. *If X is path-connected, then X is connected.*

Proof. Assume X is path-connected. Assume $X = A \cup B$ is a separation of X . Let $f: I \rightarrow X$ be a path in X . Then $f(I)$ is connected as a continuous image of a connected set. Therefore, $f(I)$ lies entirely either in A or in B . Therefore, there is no path in X joining a point in A to a point in B . This contradicts the assumption that X is path-connected. $\square$

Example 4.4. The subset

$$\{(x, \sin \frac{1}{x}) \mid 0 < x \leq \frac{1}{2\pi}\} \cup \{(0, y) \mid -1 \leq y \leq 1\}$$

of $\mathbb{R}^2$ is connected but not path-connected. This subset is called the *topologist's sine curve*.

Theorem 4.5. *Let X be a topological space. Define a relation $\sim$ on X by setting $a \sim b$ if there is a path in X from a to b . Then $\sim$ is an equivalence relation.*

Proof. Exercise. □

Definition 4.6. The equivalence classes of X under the relation $\sim$ in Theorem 4.5 are called the *path components* of X .

Definition 4.7. Let $\pi_0(X)$ denote the set of path components of X . If $f: X \rightarrow Y$ is continuous, define

$$\pi_0(f): \pi_0(X) \rightarrow \pi_0(Y)$$

to be the function taking a path component C of X to the unique path component of Y containing $f(C)$.

Theorem 4.8. $\pi_0: \text{Top} \rightarrow \text{Sets}$ is a functor. If $f \simeq g$, then $\pi_0(f) = \pi_0(g)$.

Proof. Clearly, π_0 preserves the identity and the composition. Therefore, π_0 is a functor.

Let $f, g: X \rightarrow Y$ and assume $F: f \simeq g$. Let C be a path component of X . Then $C \times I$ is path-connected. Since F is continuous, also $F(C \times I)$ is path-connected. Now

$$f(C) = F(C \times \{0\}) \subset F(C \times I)$$

and

$$g(C) = F(C \times \{1\}) \subset F(C \times I).$$

Thus the unique path component of Y containing $F(C \times I)$ contains both $f(C)$ and $g(C)$. Hence $\pi_0(f) = \pi_0(g)$. □

Corollary 4.9. *If X and Y have the same homotopy type, then they have the same number of path components.*

Proof. Let $f: X \rightarrow Y$ and $g: Y \rightarrow X$ and assume that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$. Then

$$\pi_0(g) \circ \pi_0(f) = \pi_0(g \circ f) = \pi_0(1_X) = 1_{\pi_0(X)},$$

and

$$\pi_0(f) \circ \pi_0(g) = \pi_0(f \circ g) = \pi_0(1_Y) = 1_{\pi_0(Y)}.$$

Thus $\pi_0(f)$ is a bijection. □

Definition 4.10. A topological space X is called *locally path-connected*, if the following holds: For every $x \in X$ and for every open neighborhood U of x there is an open subset V of X with $x \in V \subset U$ such that any two points in V can be joined by a path in U .

Example 4.11. Let X be the set

$$\{(x, \sin \frac{1}{x}) \mid 0 < x \leq \frac{1}{2\pi}\} \cup \{(0, y) \mid -1 \leq y \leq 1\} \cup A,$$

where A is the line segment joining the points $(0, 1)$ and $(\frac{1}{2\pi}, 0)$. Then X is path-connected but not locally path-connected.

Theorem 4.12. *A topological space X is locally path-connected if and only if the path components of open subsets are open. In particular, if X is locally path-connected, then its path components are open.*

Proof. Assume first that X is locally path-connected. Let U be an open subset of X and let C be a path component of U . Let $x \in C$. Then there is an open subset V of X such that $x \in V \subset U$ and every point in V can be joined to x by a path in U . Thus every point in V is in the same path component as x . Hence $V \subset C$. Therefore, C is open.

Assume then that the path components of open subsets of X are open. Let U be an open subset of X , let $x \in U$. Let V be the path component of x in U . Then V is open and it follows that X is locally path-connected. $\square$

Corollary 4.13. *A topological space X is locally path-connected if and only if, for every $x \in X$ and for every open neighborhood U of x , there is an open path-connected V with $x \in V \subset U$.*

Proof. If X is locally path-connected, then one can choose V to be the path component of U containing x . The converse is clear. $\square$

Corollary 4.14. *Let X be a locally path-connected topological space. Then the components of every open set coincide with its path components. In particular, the components of X coincide with the path components of X .*

Proof. Let U be an open subset of X and let C be a component of U . Let $\{A_j \mid j \in J\}$ be the path components of C . Then C is a disjoint union of the A_j . By Theorem 4.12, the A_j are open in X . Therefore, they are open in C . Since the complement of any A_j in C is the union of the A_i , $i \neq j$, it follows that the A_j are closed in C . Since C is connected, it follows that it must have exactly one path component. $\square$

Corollary 4.15. *If X is connected and locally path-connected, then X is path-connected.*

5. SIMPLEXES AND AFFINE SPACES

Definition 5.1. Let $A \subset \mathbb{R}^n$. If for all $x, x' \in A$, $x \neq x'$, the line determined by x and x' is contained in A , we call A an *affine set*.

Remark 5.2. Notice that:

- (1) The empty set and one-point subsets are affine.
- (2) All affine sets are convex.

Theorem 5.3. Let X_j , $j \in J$, be affine (or convex) subsets of $\mathbb{R}^n$. Then also $\bigcap_{j \in J} X_j$ is affine (or convex). $\square$

Let $X \subset \mathbb{R}^n$. The *affine* (or *convex*) *hull* of X is the intersection of all affine (or convex) subsets of $\mathbb{R}^n$ containing X . We also say that the affine (or convex) hull of X is the *affine* (or *convex*) *set spanned by* X .

We use the notation $[X]$ for the convex hull of X .

Definition 5.4. Let $p_0, \dots, p_m \in \mathbb{R}^n$. An *affine combination* of $p_0, \dots, p_m$ is a point x with

$$x = t_0 p_0 + \dots + t_m p_m,$$

where $\sum_{i=0}^m t_i = 1$. A *convex combination* is an affine combination such that $t_i \geq 0$, for every i .

Theorem 5.5. Let $p_0, \dots, p_m \in \mathbb{R}^n$. The convex hull $[p_0, \dots, p_m]$ of the set $\{p_0, \dots, p_m\}$ is the set of all convex combinations of $p_0, \dots, p_m$.

Proof. Let S be the set of all convex combinations of $p_0, \dots, p_m$.

We first show that $[p_0, \dots, p_m] \subset S$. It suffices to show that S is convex and $p_0, \dots, p_m \in S$. Let $j \in \{0, \dots, m\}$. Set $t_j = 1$, $t_i = 0$, for $i \neq j$. Then $p_j = \sum_{i=0}^m t_i p_i \in S$. Thus $p_0, \dots, p_m \in S$.

Let then $\alpha = \sum a_i p_i$ and $\beta = \sum b_i p_i$, where $a_i, b_i \geq 0$, $\sum a_i = 1$ and $\sum b_i = 1$. Let $t \in I$. Then

$$\begin{aligned} t\alpha + (1-t)\beta &= t \sum_{i=0}^m a_i p_i + (1-t) \sum_{i=0}^m b_i p_i \\ &= \sum_{i=0}^m (ta_i + (1-t)b_i) p_i \in S, \end{aligned}$$

since $ta_i + (1-t)b_i \geq 0$, for every i and

$$\begin{aligned} \sum_{i=0}^m (ta_i + (1-t)b_i) &= t \sum_{i=0}^m a_i + (1-t) \sum_{i=0}^m b_i \\ &= t + (1-t) = 1. \end{aligned}$$

It follows that S is convex.

We then show that $S \subset [p_0, \dots, p_m]$. In order to do that we show that $S \subset X$ for any convex subset X of $\mathbb{R}^n$ containing $\{p_0, \dots, p_m\}$. The proof is done by induction on m :

First, let $m = 0$. Then $S = \{p_0\}$ and we are done.

Let then $m \geq 0$. Let $t_i \geq 0$ for every i and assume $\sum t_i = 1$. We may assume that $t_0 \neq 1$. (If $t_0 = 1$, then $\sum t_i p_i = p_0 \in X$.) Let $p = \sum_{i=0}^m t_i p_i$. Then the convex combination

$$q = \frac{t_1}{1-t_0} p_1 + \dots + \frac{t_m}{1-t_0} p_m \in X$$

by induction. Thus $p = t_0 p_0 + (1-t_0)q \in X$, since X is convex. $\square$

The proof of the following corollary is similar to the proof of Theorem 5.5.

Corollary 5.6. *The affine set spanned by $\{p_0, \dots, p_m\}$ is the set of all affine combinations of $p_0, \dots, p_m$. $\square$*

Definition 5.7. An ordered set $\{p_0, \dots, p_m\} \subset \mathbb{R}^n$ is called *affine independent*, if $\{p_1 - p_0, \dots, p_m - p_0\}$ is a linearly independent subset of $\mathbb{R}^n$.

Remark 5.8. Notice the following:

- (1) The set $\{p_0\}$ is affine independent for $p_0 \neq 0$.
- (2) The empty set is affine independent.
- (3) The set $\{p_0, p_1\}$ is affine independent if $p_0 - p_1 \neq 0$, i.e., if $p_1 \neq p_0$.
- (4) A linearly independent set is affine independent.
- (5) Let $\{p_1, \dots, p_m\}$ be linearly independent. Then $\{0, p_1, \dots, p_m\}$ is affine independent but not linearly independent.

Theorem 5.9. *Let $\{p_0, p_1, \dots, p_m\}$ be an ordered set of points in $\mathbb{R}^n$. The following conditions are equivalent:*

- (1) *The set $\{p_0, \dots, p_m\}$ is affine independent.*
- (2) *If $\{s_0, \dots, s_m\} \subset \mathbb{R}$ satisfies $\sum_{i=0}^m s_i p_i = 0$ and $\sum_{i=0}^m s_i = 0$, then $s_0 = \dots = s_m = 0$.*
- (3) *Every element x of the affine set spanned by $\{p_0, \dots, p_m\}$ has a unique expression as an affine combination*

$$x = \sum_{i=0}^m t_i p_i, \text{ where } \sum_{i=0}^m t_i = 1.$$

Proof. We show that (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2): Assume $\sum s_i = 0$ and $\sum s_i p_i = 0$. Then

$$\begin{aligned} 0 &= \sum_{i=0}^m s_i p_i = \sum_{i=0}^m s_i p_i - \left(\sum_{i=0}^m s_i\right) p_0 \\ &= \sum_{i=0}^m s_i (p_i - p_0) = \sum_{i=1}^m s_i (p_i - p_0), \end{aligned}$$

since $p_0 - p_0 = 0$. Since the set $\{p_0, \dots, p_m\}$ is affine independent, it follows that the set $\{p_1 - p_0, \dots, p_m - p_0\}$ is linearly independent. Thus $s_i = 0$ for every $i \in \{1, \dots, m\}$. Since $\sum_{i=0}^m s_i = 0$, must be $s_0 = 0$ as well.

(2) $\Rightarrow$ (3): Let A denote the affine set spanned by $\{p_0, \dots, p_m\}$. Let $x \in A$. By Corollary 5.6,

$$x = \sum_{i=0}^m t_i p_i, \text{ where } \sum_{i=0}^m t_i = 1.$$

Assume that also $x = \sum_{i=0}^m t'_i p_i$, where $\sum_{i=0}^m t'_i = 1$. Then

$$0 = \sum_{i=0}^m (t_i - t'_i) p_i.$$

Now, $\sum_{i=0}^m (t_i - t'_i) = \sum_{i=0}^m t_i - \sum_{i=0}^m t'_i = 1 - 1 = 0$. By Condition (2), $t_i - t'_i = 0$, for all i , i.e., $t_i = t'_i$, for all i .

(3) $\Rightarrow$ (1): If $m = 0$, there is nothing to prove. Therefore, assume $m > 0$. Assume that each $x \in A$ has a unique expression as an affine combination of $p_0, \dots, p_m$. Let's make a counter assumption, by assuming that $\{p_0, \dots, p_m\}$ is not affine independent. Thus $\{p_1 - p_0, \dots, p_m - p_0\}$ is linearly dependent. It follows that there are $r_i \in \mathbb{R}$, not all of them 0, such that

$$0 = \sum_{i=1}^m r_i(p_i - p_0). \quad (*)$$

Assume $r_j \neq 0$. By multiplying $(*)$ by $\frac{1}{r_j}$ if necessary, we may assume that $r_j = 1$. There now are two different ways to write p_j as an affine combination of $p_0, \dots, p_m$:

$$p_j = 1 \cdot p_j$$

and

$$p_j = - \sum_{i \neq j} r_i(p_i - p_0) + p_0 = - \sum_{i \neq j} r_i p_i + (1 + \sum_{i \neq j} r_i) p_0.$$

This is a contradiction. Thus $\{p_0, \dots, p_m\}$ is affine independent. $\square$

We obtain the following corollaries:

Corollary 5.10. *Affine independence is a property of the set $\{p_0, \dots, p_m\}$ that is independent of the ordering.* $\square$

Corollary 5.11. *Let A be the affine set in $\mathbb{R}^n$ spanned by an affine independent set $\{p_0, \dots, p_m\}$. Then A is of the form*

$$A = x_0 + V,$$

where $x_0 \in \mathbb{R}^n$ and V is an m -dimensional vector subspace of $\mathbb{R}^n$.

Proof. Let V be the vector subspace of $\mathbb{R}^n$ whose basis is $\{p_1 - p_0, \dots, p_m - p_0\}$. Choose $x_0 = p_0$. $\square$

Definition 5.12. A set $\{a_1, \dots, a_k\}$ of points in $\mathbb{R}^n$ is in *general position*, if every subset of it consisting of $n + 1$ points is affine independent.

Remark 5.13. Assume $\{a_1, \dots, a_k\}$ is in general position.

- (1) Assume $n = 1$. Then every pair $\{a_i, a_j\}$ is affine independent, i.e., $a_i \neq a_j$ for $i \neq j$.
- (2) Assume $n = 2$. A three-point set $\{a_i, a_j, a_k\}$ is affine independent, if and only if $\{a_j - a_i, a_k - a_i\}$ is linearly independent. This means that no three points a_i, a_j, a_k can lie in a single straight line.
- (3) Assume $n = 3$. No four points of $\{a_0, \dots, a_m\}$ can lie in a single plane.

Definition 5.14. Let $\{p_0, \dots, p_m\}$ be an affine independent subset of $\mathbb{R}^n$. Let A be the affine set spanned by $\{p_0, \dots, p_m\}$. Let $x \in A$ and let $(t_0, \dots, t_m)$ be the unique $(m + 1)$ -tuple with $\sum_{i=0}^m t_i = 1$ and $x = \sum_{i=0}^m t_i p_i$ (such a tuple exists by Theorem 5.9). The numbers $t_0, \dots, t_m$ are called the *barycentric coordinates* of x (relative to the ordered set $\{p_0, \dots, p_m\}$).

Definition 5.15. Let $\{p_0, \dots, p_m\}$ be an affine independent subset of $\mathbb{R}^n$. The convex set $[p_0, \dots, p_m]$ spanned by $\{p_0, \dots, p_m\}$ is called the (affine) m -simplex with vertices $p_0, \dots, p_m$.

Theorem 5.16. Let $\{p_0, \dots, p_m\}$ be an affine independent set. Then every $x \in [p_0, \dots, p_m]$ has a unique expression of the form

$$x = \sum_{i=0}^m t_i p_i,$$

where $\sum_{i=0}^m t_i = 1$ and $t_i \geq 0$, for every i .

Proof. By Theorem 5.5, every $x \in [p_0, \dots, p_m]$ has an expression of such form. By Theorem 5.9, the expression is unique. $\square$

Definition 5.17. Let $\{p_0, \dots, p_m\}$ be an affine independent set. The *barycenter* of $[p_0, \dots, p_m]$ is

$$\frac{1}{m+1}(p_0 + \dots + p_m).$$

Example 5.18. The set $[p_0]$ is a 0-simplex consisting of one point, which is its own barycenter.

The 1-simplex

$$[p_0, p_1] = \{tp_0 + (1-t)p_1 \mid t \in I\}$$

is a line segment. The barycenter of $[p_0, p_1]$ is $\frac{1}{2}(p_0 + p_1)$, i.e., the midpoint of $[p_0, p_1]$.

The 2-simplex $[p_0, p_1, p_2]$ is a triangle (with interior) with vertices p_0, p_1 and p_2 . The barycenter of $[p_0, p_1, p_2]$ is $\frac{1}{3}(p_0 + p_1 + p_2)$, i.e., the center of gravity of $[p_0, p_1, p_2]$. The edges of $[p_0, p_1, p_2]$ are $[p_0, p_1]$, $[p_0, p_2]$ and $[p_1, p_2]$. Consider the edge $[p_0, p_1]$. The barycentric coordinates of $x \in [p_0, p_1]$ are of the form $(t, 1-t, 0)$. Generally, $x \in [p_0, p_1, p_2]$ lies on an edge if and only if one of its barycentric coordinates is 0.

The 3-simplex $[p_0, p_1, p_2, p_3]$ is a solid tetrahedron with vertices p_0, p_1, p_2 and p_3 . The triangular face opposite of p_i consists of those points of $[p_0, p_1, p_2, p_3]$ whose i^{th} barycentric coordinate equals 0.

Example 5.19. (Standard n -simplex) Let the standard basis vector for $\mathbb{R}^{n+1}$ be $e_0, \dots, e_n$. Then e_i is the vector in $\mathbb{R}^{n+1}$ whose $(i+1)^{\text{st}}$ cartesian coordinate equals 1, and all other coordinates are 0. The set $\{e_0, \dots, e_n\}$ is linearly independent, hence also affine independent. The n -simplex $[e_0, \dots, e_n]$ is called the *standard n -simplex* and denoted by Δ^n . It consists of all convex combinations $x = \sum t_i e_i$. In this case, barycentric and cartesian coordinates coincide.

Definition 5.20. Let $[p_0, \dots, p_m]$ be an m -simplex. The *face opposite* p_i is the $(m-1)$ -simplex

$$[p_0, \dots, \hat{p}_i, \dots, p_m] = \left\{ \sum t_j p_j \mid t_j \geq 0, t_i = 0 \right\}.$$

The *boundary* of $[p_0, \dots, p_m]$ is the union of its faces.

In the previous definition, the notation $[p_0, \dots, \hat{p}_i, \dots, p_m]$ means that the vertex $\hat{p}_i$ is deleted.

An m -simplex has $m + 1$ faces.

Let $0 \leq k \leq m - 1$. A k -simplex spanned by $k + 1$ of the vertices $p_0, \dots, p_m$ is called a k -face of $[p_0, \dots, p_m]$. Thus the faces defined as in Definition 5.20 can be called $(m - 1)$ -faces.

The *diameter* of a subset S of a euclidean space is defined to be

$$\text{diam}(S) = \sup\{\|u - v\| \mid u, v \in S\}.$$

The following theorem will be needed later:

Theorem 5.21. *Let S be the n -simplex $[p_0, \dots, p_n]$. Then:*

- (1) *If $u, v \in S$, then $\|u - v\| \leq \sup_i \|u - p_i\|$.*
- (2) *The diameter of S equals $\sup_{i,j} \|p_i - p_j\|$.*
- (3) *If b is the barycenter of S , then*

$$\|b - p_i\| \leq \frac{n}{n+1} \text{diam}(S).$$

Proof. Write $v = \sum t_i p_i$, where $t_i \geq 0$ and $\sum t_i = 1$. Then

$$\begin{aligned} \|u - v\| &= \|u - \sum t_i p_i\| = \|(\sum t_i)u - \sum t_i p_i\| \\ &= \|\sum t_i(u - p_i)\| \leq \sum t_i \|u - p_i\| \\ &\leq \sum t_i \sup_j \|u - p_j\| = \sup_i \|u - p_i\|. \end{aligned}$$

This proves the first claim.

The second claim follows since,

$$\begin{aligned} \|u - v\| &\leq \sup_i \|u - p_i\| \leq \sup_i (\sup_j \|p_j - p_i\|) \\ &= \sup_{i,j} \|p_j - p_i\|, \end{aligned}$$

for every $u, v \in S$.

By definition, $b = \frac{1}{n+1} \sum p_i$. Thus

$$\begin{aligned} \|b - p_i\| &= \left\| \sum_{j=0}^n \frac{1}{n+1} p_j - p_i \right\| = \left\| \sum_{j=0}^n \frac{1}{n+1} p_j - \left(\sum_{j=0}^n \frac{1}{n+1} \right) p_i \right\| \\ &= \left\| \sum_{j=0}^n \frac{1}{n+1} (p_j - p_i) \right\| \leq \sum_{j=0}^n \frac{1}{n+1} \|p_j - p_i\| \\ &= \frac{1}{n+1} \sum_{j=0}^n \|p_j - p_i\| \\ &\leq \frac{n}{n+1} \sup_{i,j} \|p_j - p_i\| \quad (\text{since } \|p_j - p_i\| = 0 \text{ if } i = j) \\ &= \frac{n}{n+1} \text{diam}(S). \end{aligned}$$

This proves the last claim. $\square$

Definition 5.22. Let A be the set spanned by the affine independent set $\{p_0, \dots, p_m\}$. Let $k \geq 1$. An *affine map* $T: A \rightarrow \mathbb{R}^k$ is a function satisfying

$$T\left(\sum t_j p_j\right) = \sum t_j T(p_j)$$

whenever $\sum t_j = 1$.

Affine maps preserve affine combinations, hence also convex combinations. Also the restriction of T to the convex hull $[p_0, \dots, p_m]$ is called an affine map. An affine map is determined by its values on an affine independent subset. Thus its restriction to a simplex is determined by its values on the vertices.

Theorem 5.23. Let $[p_0, \dots, p_m]$ be an m -simplex and let $[q_0, \dots, q_n]$ be an n -simplex. Let $f: \{p_0, \dots, p_m\} \rightarrow [q_0, \dots, q_n]$ be any function. Then there is a unique affine map $T: [p_0, \dots, p_m] \rightarrow [q_0, \dots, q_n]$ satisfying $T(p_i) = f(p_i)$, for every $i \in \{0, \dots, m\}$.

Proof. Define $T(\sum t_i p_i) = \sum t_i f(p_i)$ for convex combinations $\sum t_i p_i$. Uniqueness is clear. $\square$

The proof of the following theorem is left as an exercise:

Theorem 5.24. Every affine map is continuous. $\square$

Theorem 5.25. (Radon's theorem, 1905) Let $x_1, \dots, x_{n+2}$ be distinct points in $\mathbb{R}^n$. Then the set $\{x_1, \dots, x_{n+2}\}$ can be partitioned in two subsets S and T such that $[S] \cap [T] \neq \emptyset$, where $[S]$ and $[T]$ denote the convex hulls of S and T , respectively.

Proof. Consider the following equations for solving the unknown real numbers $\alpha_1, \dots, \alpha_{n+2}$:

$$\begin{cases} \alpha_1 x_1 + \dots + \alpha_{n+2} x_{n+2} = 0 \\ \alpha_1 + \dots + \alpha_{n+2} = 0 \end{cases}$$

Since there is one equation for each coordinate and one more equation, there are altogether $n+1$ equations. Since there are $n+1$ equations and $n+2$ unknowns, it follows that the system has a nontrivial solution $(\alpha_1, \dots, \alpha_{n+2})$. Without loss of generality, we may assume that

$$0 > \alpha_1 \leq \dots \leq \alpha_k \leq 0 \leq \alpha_{k+1} \leq \dots \leq \alpha_{n+2} > 0,$$

for some $k \in \{1, \dots, n+1\}$. Then

$$-\alpha_1 - \dots - \alpha_k = \alpha_{k+1} + \dots + \alpha_{n+2} > 0.$$

Thus

$$\begin{aligned} p &= \frac{-\alpha_1}{-(\alpha_1 + \dots + \alpha_k)} x_1 + \dots + \frac{-\alpha_k}{-(\alpha_1 + \dots + \alpha_k)} x_k \\ &= \frac{\alpha_{k+1}}{(\alpha_{k+1} + \dots + \alpha_{n+2})} x_{k+1} + \dots + \frac{\alpha_{n+2}}{(\alpha_{k+1} + \dots + \alpha_{n+2})} x_{n+2} = q. \end{aligned}$$

Let $S = \{x_1, \dots, x_k\}$ and $T = \{x_{k+1}, \dots, x_{n+2}\}$. Then $p \in [S]$ and $q \in [T]$. Since $p = q$, it follows that $[S] \cap [T] \neq \emptyset$. $\square$

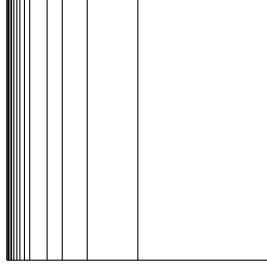
Radon's theorem can be reformulated by using simplexes: For every affine map $f: \Delta^{n+1} \rightarrow \mathbb{R}^n$, there exist two faces S and T of Δ^{n+1} , $S \cap T = \emptyset$, such that $f(S) \cap f(T) \neq \emptyset$.

6. ON RETRACTS, DEFORMATION RETRACTS AND STRONG DEFORMATION RETRACTS

Let

$$X = \{(\frac{1}{n}, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1, n \in \mathbb{N}\} \cup (\{0\} \times [0, 1]) \cup ([0, 1] \times \{0\}).$$

be equipped with the relative topology from $\mathbb{R}^2$. The space X is called the *comb space*. The comb space is a good space for examples and counterexamples. For example, it is connected but not locally connected, and path-connected but not locally path-connected.



The comb space X

Lemma 6.1. *The comb space is contractible.*

Proof. Define

$$H: X \times I \rightarrow X, \quad ((x, y), t) \mapsto \begin{cases} (x, (1 - 2t)y), & \text{if } 0 \leq t \leq \frac{1}{2}, \\ ((2 - 2t)x, 0), & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

For $t = \frac{1}{2}$,

$$(x, (1 - 2 \cdot \frac{1}{2})y) = (x, 0) = ((2 - 2 \cdot \frac{1}{2})x, 0).$$

It now follows from the first gluing lemma that H is continuous. For $t = 0$, $H((x, y), 0) = (x, y)$, and for $t = 1$, $H((x, y), 1) = (0, 0)$. It follows that the identity map of X is null-homotopic, and hence that X is contractible. $\square$

Definition 6.2. Let X be a topological space, let $A \subset X$ and let $i: A \hookrightarrow X$ be the inclusion. If there is a continuous map $r: X \rightarrow A$ such that $r \circ i = 1_A$, then A is called a *retract* of X . In this case the map r is called a *retraction* of X to A .

Example 6.3. Let $Y = [0, 1] \times [0, 1]$ and let X be the comb space. Then $X \subset Y$ and both X and Y are contractible, i.e., they have the same homotopy type. We show that X is not a retract of Y : Let $r: Y \rightarrow X$ be a continuous map. Let $(0, y) \in X$, $y \neq 0$. Assume $r(0, y) = (0, y)$. Let U be a neighborhood of $(0, y)$ in X , we may assume that U is small enough so that it does not intersect the X -axis. Since r is continuous at $(0, y)$, it follows that $r(V) \subset U$, for arbitrarily small neighborhoods V of $(0, y)$ in Y . Since Y is locally connected, we may assume

that V is connected. Thus also $r(V)$ is connected. Therefore, $r(V)$ must lie on the y -axis. Thus the restriction $r|_X \neq i: X \hookrightarrow Y$, and it follows that r can not be a retraction.

Definition 6.4. Let X be a topological space and let $A \subset X$. Assume there is a continuous map $F: X \times I \rightarrow X$ with the following properties:

- (1) $F(x, 0) = x$, for all $x \in X$,
- (2) $F(x, 1) \in A$, for all $x \in X$,
- (3) $F(a, 1) = a$, for all $a \in A$.

Then A is called a *deformation retract* of X . If, in addition, $F(a, t) = a$ for all $a \in A$ and for all $t \in I$, then A is called a *strong deformation retract* of X .

Equivalently, A is a deformation retract of X , if there is a continuous $r: X \rightarrow A$ with $r \circ i = 1_A$ and $i \circ r \simeq 1_X$, where $i: A \hookrightarrow X$ is the inclusion.

The following follows immediately from the definition:

Theorem 6.5. *If A is a deformation retract of X , then A and X have the same homotopy type.* $\square$

Corollary 6.6. *The circle $\mathbb{S}^1$ is a deformation retract of $\mathbb{C} \setminus \{0\}$. The spaces $\mathbb{S}^1$ and $\mathbb{C} \setminus \{0\}$ have the same homotopy type.*

Proof. Let

$$F: (\mathbb{C} \setminus \{0\}) \times I \rightarrow \mathbb{C} \setminus \{0\}, \quad (z, t) \mapsto (1 - t)z + \frac{t}{\|z\|}z.$$

Then F satisfies conditions (1) - (3) above. Thus $\mathbb{S}^1$ is a deformation retract of $\mathbb{C} \setminus \{0\}$. $\square$

Remark 6.7. Some authors, like A. Hatcher [1], call a deformation retract what we call a strong deformation retract.

Example 6.8. Let $X = \mathbb{S}^1$ and $A = \{(x, y) \in X \mid x \geq 0\}$. Let $r: X \rightarrow A$, $(x, y) \mapsto (|x|, y)$. Then r is a retraction. However, X and A do not have the same homotopy type, since A is contractible but X is not, as will be proved later. Thus A is not a deformation retract of X .

Example 6.9. Let

$$X = \left\{ \left(\frac{1}{n}, y \right) \in \mathbb{R}^2 \mid 0 \leq y \leq 1, n \in \mathbb{N} \right\} \cup (\{0\} \times [0, 1]) \cup ([0, 1] \times \{0\}).$$

be the comb space. We show that the one-point space $\{(0, 1)\}$ is a deformation retract of X but not a strong deformation retract of X . Notice that it does not matter on which finite interval a homotopy is defined. Thus the homotopy

$$H: X \times [0, 2] \rightarrow X, \quad ((x, y), t) \mapsto \begin{cases} (x, (1 - 2t)y), & \text{if } 0 \leq t \leq \frac{1}{2}, \\ ((2 - 2t)x, 0), & \text{if } \frac{1}{2} \leq t \leq 1, \\ (0, t - 1), & \text{if } 1 \leq t \leq 2, \end{cases}$$

deformation retracts X to $\{(0, 1)\}$.

Assume $F: X \times I \rightarrow X$ is a homotopy that strongly deformation retracts X to the point $(0, 1)$. Then

- (1) $F(\{(0, 1)\} \times I) = \{(0, 1)\}$,
- (2) $F((\frac{1}{n}, 1), 1) = (0, 1)$, for every $n \in \mathbb{N}$,
- (3) $F((\frac{1}{n}, 1), 0) = (\frac{1}{n}, 1)$, for every $n \in \mathbb{N}$.

Let U be a neighborhood of $(0, 1)$. Since F is continuous, the set $\{(0, 1)\} \times I$ has a neighborhood N such that $F(N) \subset U$. By the Tube Lemma (Lemma 3.17), we may assume that N is of the form $N = V \times I$, where V is a neighborhood of $(0, 1)$. Now, there exist $m \in \mathbb{N}$ such that for every $n \geq m$, $(\frac{1}{n}, 1) \in V$. Thus $F(\{(\frac{1}{n}, 1)\} \times I) \subset U$. Since $F((\frac{1}{n}, 1), 0) = (\frac{1}{n}, 1)$, $F((\frac{1}{n}, 1), 1) = (0, 1)$ and $F(\{(\frac{1}{n}, 1)\} \times I)$ is connected, this is impossible if U is an arbitrarily small neighborhood of $(0, 1)$: a path from $(\frac{1}{n}, 1)$ to $(0, 1)$ must go through the x -axis and can not stay in an arbitrarily small neighborhood U . It follows that $\{(0, 1)\}$ cannot be a strong deformation retract of X .

7. THE FUNDAMENTAL GROUPOID

Definition 7.1. Let X be a topological space and let $f, g: I \rightarrow X$ be paths with $f(1) = g(0)$. Define a path $f * g: I \rightarrow X$ by

$$(f * g)(t) = \begin{cases} f(2t), & \text{if } 0 \leq t \leq \frac{1}{2}, \\ g(2t - 1), & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

The path $f * g$ is called the *concatenation* of f and g .

Notice that $f * g$ is continuous by the first gluing lemma.

Let $[f]$ denote the homotopy class of the path f . Define an operation among the homotopy classes by setting

$$[f][g] = [f * g].$$

Lemma 7.2. *Let X and Y be topological spaces. Assume X is contractible and Y is path-connected. Then any two continuous maps $f, g: X \rightarrow Y$ are homotopic (and each is nullhomotopic).*

Proof. Since X is contractible, there exist $x_0 \in X$ and a homotopy $F: 1_X \simeq c_{x_0}$, where c_{x_0} denotes the constant map at x_0 . Then $f \circ F: f \simeq c_{f(x_0)}$ and $g \circ F: g \simeq c_{g(x_0)}$. Since Y is path-connected, there is a path $h: I \rightarrow Y$, with $h(0) = f(x_0)$ and $h(1) = g(x_0)$. Let $\text{pr}: X \times I \rightarrow I$ be the projection. Then

$$H = h \circ \text{pr}: X \times I \rightarrow Y, \quad (x, t) \mapsto h(t),$$

is a homotopy from $c_{f(x_0)}$ to $c_{g(x_0)}$. Thus

$$f \simeq c_{f(x_0)} \simeq c_{g(x_0)} \simeq g.$$

□

Assume X is path-connected. Since I is contractible, Lemma 7.2 implies that all paths $I \rightarrow X$ are homotopic. Therefore, there is only one homotopy class of maps $I \rightarrow X$.

Definition 7.3. Let $A \subset X$ and let $f_0, f_1: X \rightarrow Y$ be continuous maps. Assume $f_0|_A = f_1|_A$. If there is a continuous map

$$F: X \times I \rightarrow Y, \text{ with } F: f_0 \simeq f_1,$$

and

$$F(a, t) = f_0(a) = f_1(a), \text{ for every } a \in A, t \in I,$$

we write

$$F: f_0 \simeq f_1 \text{ rel } A.$$

The homotopy F is called a *homotopy rel A* or a *relative homotopy*.

For $A = \emptyset$ we get our earlier definition of a homotopy, which is also called a *free homotopy*.

Exercise 7.4. Let $A \subset X$. Show that homotopy rel A is an equivalence relation on the set of continuous maps $X \rightarrow Y$.

Definition 7.5. Let $\dot{I} = \{0, 1\}$. The equivalence class of a path $f: I \rightarrow X \text{ rel } \dot{I}$ is called the *path class* of f and denoted by $[f]$.

Theorem 7.6. Let f_0, f_1, g_0, g_1 be paths in X . Assume

$$f_0 \simeq f_1 \text{ rel } \dot{I} \text{ and } g_0 \simeq g_1 \text{ rel } \dot{I}.$$

If $f_0(1) = f_1(1) = g_0(0) = g_1(0)$, then

$$f_0 * g_0 \simeq f_1 * g_1 \text{ rel } \dot{I}.$$

Proof. Let $F: f_0 \simeq f_1 \text{ rel } \dot{I}$ and $G: g_0 \simeq g_1 \text{ rel } \dot{I}$. Let

$$H: I \times I \rightarrow X, (t, s) \mapsto \begin{cases} F(2t, s), & \text{if } 0 \leq t \leq \frac{1}{2}, \\ G(2t - 1, s), & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Since $F(1, s) = G(0, s)$ for every $s \in I$, it follows that H is well-defined. By the first gluing lemma H is continuous. Thus $H: f_0 * g_0 \simeq f_1 * g_1 \text{ rel } \dot{I}$. $\square$

Definition 7.7. Let $x_0, x_1 \in X$ and let $f: I \rightarrow X$ be a path, $f(0) = x_0$, $f(1) = x_1$. Then x_0 is called the *origin* of f ($x_0 = \alpha(f)$) and x_1 is called the *end* of f ($x_1 = \omega(f)$). A path f is *closed* at x_0 , if $\alpha(f) = \omega(f)$.

Let $f, g: I \rightarrow X$ be paths, $f \simeq g \text{ rel } \dot{I}$. Then $\alpha(f) = \alpha(g)$ and $\omega(f) = \omega(g)$. Thus it makes sense to speak about the origin and end of a path class, those are denoted by $\alpha[f]$ and $\omega[f]$, respectively.

Definition 7.8. Let $p \in X$. The constant function $i_p: I \rightarrow X, t \mapsto p$, is called the *constant path* at p . The *inverse path* of a path $f: I \rightarrow X$ is the path $f^{-1}: I \rightarrow X, t \mapsto f(1 - t)$.

Theorem 7.9. Let X be a topological space. The set of all path classes in X forms an algebraic system called a *groupoid* under the operation $[f][g] = [f * g]$ (not always defined) satisfying the following properties:

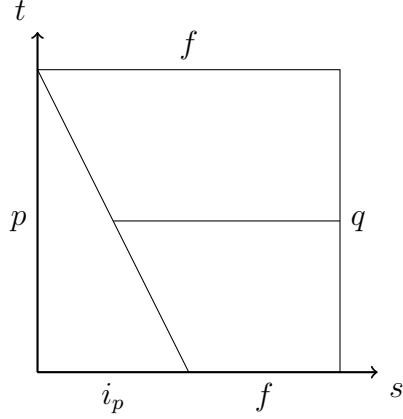
- (1) Each path class $[f]$ has an origin $\alpha[f] = p \in X$ and an end $\omega[f] = q \in X$ and

$$[i_p][f] = [f] = [f][i_q].$$

- (2) *Associativity holds whenever possible.*
 (3) *If $p = \alpha[f]$ and $q = \omega[f]$, then*

$$[f][f^{-1}] = [i_p] \text{ and } [f^{-1}][f] = [i_q].$$

Proof. We show that $i_p * f \simeq f \text{ rel } \dot{I}$. Similarly it can be shown that $f * i_q \simeq f \text{ rel } \dot{I}$. The following picture explains the construction of a suitable homotopy:



For every $t \in [0, 1)$, stretch the interval $[\frac{1-t}{2}, 1]$ to $[0, 1]$ by using an affine map

$$\theta_t: [\frac{1-t}{2}, 1] \rightarrow [0, 1], \quad s \mapsto \frac{s - (1-t)/2}{1 - (1-t)/2}.$$

Let

$$H: I \times I \rightarrow X, \quad (s, t) \mapsto \begin{cases} p, & \text{if } 2s \leq 1 - t, \\ f(\theta_t(s)), & \text{if } 2s \geq 1 - t. \end{cases}$$

It is left for the reader to check that $H: i_p * f \simeq f \text{ rel } \dot{I}$.

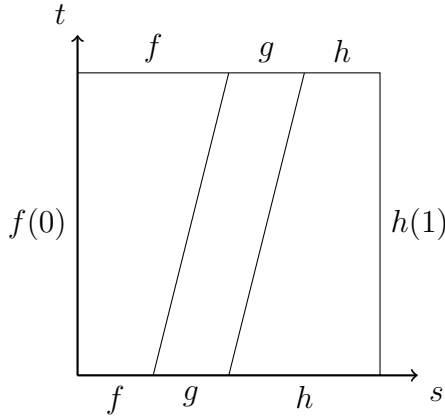
Let then $f, g, h: I \rightarrow X$. Assume $f(1) = g(0)$ and $g(1) = h(0)$. Then $(f * g) * h$ and $f * (g * h)$ are defined,

$$(f * g) * h(s) = \begin{cases} f(4s), & \text{if } 0 \leq s \leq \frac{1}{4}, \\ g(4s - 1), & \text{if } \frac{1}{4} \leq s \leq \frac{1}{2}, \\ h(2s - 1), & \text{if } \frac{1}{2} \leq s \leq 1, \end{cases}$$

and

$$f * (g * h)(s) = \begin{cases} f(2s), & \text{if } 0 \leq s \leq \frac{1}{2}, \\ g(4s - 2), & \text{if } \frac{1}{2} \leq s \leq \frac{3}{4}, \\ h(4s - 3), & \text{if } \frac{3}{4} \leq s \leq 1. \end{cases}$$

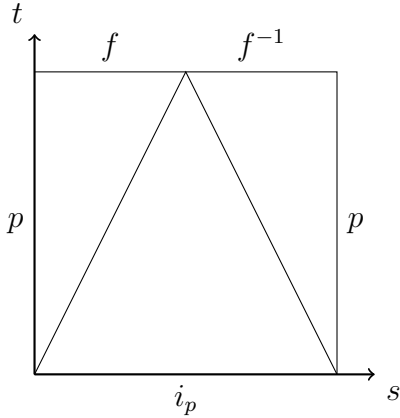
The following picture explains how to construct a homotopy $H: (f * g) * h \simeq f * (g * h) \text{ rel } \dot{I}$:



It is left to show that $i_p \simeq f * f^{-1} \text{ rel } \dot{I}$. Here

$$\begin{aligned} (f * f^{-1})(s) &= \begin{cases} f(2s), & \text{if } 0 \leq s \leq \frac{1}{2}, \\ f^{-1}(2s - 1), & \text{if } \frac{1}{2} \leq s \leq 1, \end{cases} \\ &= \begin{cases} f(2s), & \text{if } 0 \leq s \leq \frac{1}{2}, \\ f^{-1}(2 - 2s), & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases} \end{aligned}$$

The following picture will help to construct a homotopy $H: i_p \simeq f * f^{-1} \text{ rel } \dot{I}$:



Then

$$H(s, t) = \begin{cases} f(2s), & \text{if } 0 \leq s \leq \frac{t}{2}, \\ f(t), & \text{if } \frac{t}{2} \leq s \leq 1 - \frac{t}{2}, \\ f(2 - 2s), & \text{if } 1 - \frac{t}{2} \leq s \leq 1. \end{cases}$$

□

Definition 7.10. Choose $x_0 \in X$ and call it the *basepoint*. The *fundamental group* of X with basepoint x_0 is

$$\pi_1(X, x_0) = \{[f] \mid [f] \text{ is a path class in } X \text{ with } \alpha[f] = x_0 = \omega[f]\}$$

with the operation $[f][g] = [f * g]$.

Theorem 7.11. $\pi_1(X, x_0)$ is a group for each $x_0 \in X$.

□

8. THE FUNCTOR π_1

The *category of pointed spaces*, Top_* , was defined in Example 1.10. The objects of Top_* are all ordered pairs (X, x_0) where X is a topological space and $x_0 \in X$ is a basepoint of X . A morphism $f: (X, x_0) \rightarrow (Y, y_0)$ is a continuous map $f: X \rightarrow Y$ such that $f(x_0) = y_0$. Morphisms of this category are called *pointed maps* or *basepoint preserving maps*. Objects are called *pointed spaces*.

For the standard unit circle $\mathbb{S}^1$ we always choose the basepoint $1 = (1, 0)$, and for the unit interval I we choose the basepoint 0 .

Theorem 8.1. $\pi_1: \text{Top}_* \rightarrow \text{Groups}$ is a functor. If $h, k: (X, x_0) \rightarrow (Y, y_0)$ and $h \simeq k \text{ rel } \{x_0\}$, then $\pi_1(h) = \pi_1(k)$.

Proof. Define

$$\pi_1(h): \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0), \quad [f] \mapsto [h \circ f].$$

Since $h \circ f: I \rightarrow Y$ is continuous and

$$(h \circ f)(0) = h(f(0)) = h(x_0) = y_0 = h(f(1)) = (h \circ f)(1),$$

it follows that $[h \circ f] \in \pi_1(Y, y_0)$. If $f \simeq f' \text{ rel } \dot{I}$, then it follows that $h \circ f \simeq h \circ f' \text{ rel } \dot{I}$ (check this!). Thus $\pi_1(h)$ is well-defined.

Let f and g be closed paths in X at x_0 . Then

$$h \circ (f * g) = (h \circ f) * (h \circ g).$$

Thus it follows that $\pi_1(h)$ is a homomorphism.

Let $1_X: X \rightarrow X$ be the identity function. Then $\pi_1(1_X): \pi_1(X, x_0) \rightarrow \pi_1(X, x_0)$ is the identity. Also, for $h: (X, x_0) \rightarrow (Y, y_0)$ and $l: (Y, y_0) \rightarrow (Z, z_0)$, $\pi_1(l \circ h) = \pi_1(l) \circ \pi_1(h)$. Thus π_1 is a functor.

Let f be a closed path in X at x_0 . Let $h \simeq k \text{ rel } \{x_0\}$. Then $h \circ f \simeq k \circ f \text{ rel } \dot{I}$. Thus $[h \circ f] = [k \circ f]$. It follows that $\pi_1(h) = \pi_1(k)$. $\square$

We will often use the notation h_* for the homomorphism $\pi_1(h)$ induced by h .

We next define the *pointed homotopy category* hTop_* . The objects in hTop_* are the pointed spaces (X, x_0) . Morphisms $(X, x_0) \rightarrow (Y, y_0)$ are the relative homotopy classes $[f]$ of pointed maps $f: (X, x_0) \rightarrow (Y, y_0)$. Composition is defined by setting $[h \circ f] = [h] \circ [f]$ for such h and f that composing in Top_* is possible.

Theorem 8.2. Let $x_0 \in X$ and let X_0 be the path component of X containing x_0 . Then

$$\pi_1(X, x_0) \cong \pi_1(X_0, x_0).$$

Proof. Let $i: (X_0, x_0) \hookrightarrow (X, x_0)$ be the inclusion. Then i induces the homomorphism $i_*: \pi_1(X_0, x_0) \rightarrow \pi_1(X, x_0)$. Assume $[f] \in \ker i_*$. Then $i \circ f \simeq c \text{ rel } \dot{I}$, where $c: I \rightarrow X$, $t \mapsto x_0$, is the constant path at x_0 . Let $F: i \circ f \simeq c \text{ rel } \dot{I}$. Then $F(0, 0) = (i \circ f)(0) = x_0 \in X_0$, and $F(I \times I)$ is path-connected, since $I \times I$ is path-connected. Therefore, $F(I \times I) \subset X_0$. It follows that $i \circ f$ is nullhomotopic in X_0 . Consequently, i_* is injective.

Let then $f: I \rightarrow X$ be a closed path at x_0 . Then $f(I) \subset X_0$. Let $f': I \rightarrow X_0$, $f'(t) = f(t)$ for every $t \in I$. Then $i \circ f' = f$. Thus $i_*[f'] = [i \circ f'] = [f]$. Therefore, i_* is surjective, which proves the claim. $\square$

Theorem 8.3. *Let X be path-connected and let $x_0, x_1 \in X$. Then*

$$\pi_1(X, x_0) \cong \pi_1(X, x_1).$$

Proof. Since X is path-connected, there is a path γ in X from x_0 to x_1 . Let

$$\varphi: \pi_1(X, x_0) \rightarrow \pi_1(X, x_1), \quad [f] \mapsto [\gamma^{-1}][f][\gamma].$$

The homomorphism φ is an isomorphism by Theorem 7.9. The inverse of φ is given by $[g] \mapsto [\gamma][g][\gamma^{-1}]$. $\square$

Theorem 8.4. *Let (X, x_0) and (Y, y_0) be pointed spaces. Then*

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

Proof. Let $p: (X \times Y, (x_0, y_0)) \rightarrow (X, x_0)$, and $q: (X \times Y, (x_0, y_0)) \rightarrow (Y, y_0)$ be projections. Then

$$(p, q): (X \times Y, (x_0, y_0)) \rightarrow (X \times Y, (x_0, y_0)),$$

induces

$$(p_*, q_*): \pi_1(X \times Y, (x_0, y_0)) \rightarrow \pi_1(X, x_0) \times \pi_1(Y, y_0), \quad [f] \mapsto (p_*[f], q_*[f]),$$

where f is a closed path at (x_0, y_0) and $(p_*[f], q_*[f]) = ([p \circ f], [q \circ f])$. Then (p_*, q_*) is a homomorphism. We show that it is an isomorphism by finding its inverse.

Let $g: I \rightarrow X$ be a closed path in X at x_0 and let $h: I \rightarrow Y$ be a closed path in Y at y_0 . Let

$$\theta: \pi_1(X, x_0) \times \pi_1(Y, y_0) \rightarrow \pi_1(X \times Y, (x_0, y_0)), \quad ([g], [h]) \mapsto [(g, h)],$$

where

$$(g, h): I \rightarrow X \times Y, \quad t \mapsto (g(t), h(t)).$$

Then θ is the inverse of (p_*, q_*) . $\square$

Lemma 8.5. *Let $\varphi_0, \varphi_1: X \rightarrow Y$ be continuous. Assume there is a (free) homotopy $F: \varphi_0 \simeq \varphi_1$. Let $x_0 \in X$ and let λ be the path $F(x_0, \cdot)$ in Y from $\varphi_0(x_0)$ to $\varphi_1(x_0)$. Then there is a commutative diagram*

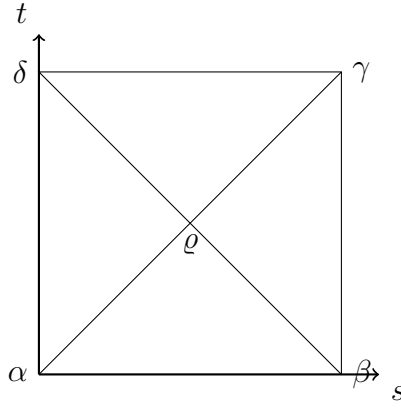
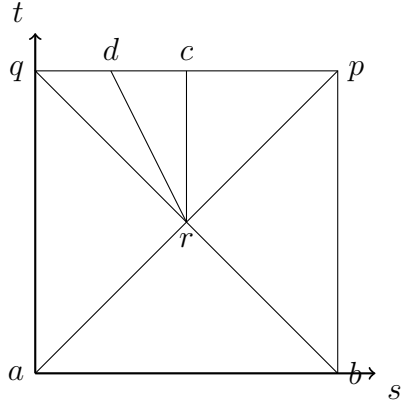
$$\begin{array}{ccc} \pi_1(X, x_0) & & \\ \varphi_{1*} \downarrow & \searrow \varphi_{0*} & \\ \pi_1(Y, \varphi_1(x_0)) & \xrightarrow[\psi]{} & \pi_1(Y, \varphi_0(x_0)) \end{array}$$

where ψ is the isomorphism $[g] \mapsto [\lambda * g * \lambda^{-1}]$.

Proof. Let $f: I \rightarrow X$ be a closed path at x_0 . Let

$$G: I \times I \rightarrow Y, \quad (s, t) \mapsto F(f(s), t).$$

Then $G: \varphi_0 \circ f \simeq \varphi_1 \circ f$. Consider the following triangulations of $I \times I$:



Define a continuous map $H: I \times I \rightarrow I \times I$: First, define H on each triangle, then use the gluing lemma to see that H is continuous. On every triangle (i.e., on 2-simplices), H will be an affine map. Thus it suffices to evaluate it on each vertex, and the maps defined on overlaps will agree automatically. Define

- (1) $H(a) = H(q) = \alpha$,
- (2) $H(b) = H(p) = \beta$,
- (3) $H(c) = \gamma$,
- (4) $H(d) = \delta$,
- (5) $H(r) = \rho$,

Then

- (1) $[a, q]$ collapses to α ,
- (2) $[b, p]$ collapses to β ,
- (3) $[q, d]$ goes to $[\alpha, \delta]$,
- (4) $[d, c]$ goes to $[\delta, \gamma]$,
- (5) $[c, p]$ goes to $[\gamma, \beta]$,
- (6) $[a, b]$ goes to $[\alpha, \beta]$.

Define

$$J = G \circ H: I \times I \rightarrow Y, \quad (s, t) \mapsto G(H(s, t)).$$

Then

$$\begin{aligned} J(s, 0) &= G(H(s, 0)) = G(s, 0) = F(f(s), 0) = (\varphi_0 \circ f)(s), \\ J(s, 1) &= G(H(s, 1)) = (\lambda * (\varphi_1 \circ f) * \lambda^{-1})(s), \\ J(0, t) &= G(H(0, t)) = G(0, 0) = F(f(0), 0) = \varphi_0(f(0)) = \varphi_0(x_0), \\ J(1, t) &= G(H(1, t)) = G(1, 0) = F(f(1), 0) = \varphi_0(f(1)) = \varphi_0(x_0). \end{aligned}$$

Thus

$$J: \varphi_0 \circ f \simeq \lambda * (\varphi_1 \circ f) * \lambda^{-1} \text{ rel } I.$$

Now,

$$\varphi_{0*}[f] = [\varphi_0 \circ f] = [\lambda * (\varphi_1 \circ f) * \lambda^{-1}],$$

and also

$$(\psi \circ \varphi_{1*})[f] = \psi(\varphi_{1*}[f]) = \psi([\varphi_1 \circ f]) = [\lambda * (\varphi_1 \circ f) * \lambda^{-1}].$$

It follows that $\varphi_{0*} = \psi \circ \varphi_{1*}$. $\square$

By Lemma 8.5, freely homotopic maps φ_0 and φ_1 may not induce the same homomorphism between fundamental groups, the induced homomorphisms differ by the isomorphism ψ .

Corollary 8.6. *Assume $\varphi_0, \varphi_1: (X, x_0) \rightarrow (Y, y_0)$ are freely homotopic. Then:*

- (1) *There is $[\lambda] \in \pi_1(Y, y_0)$ such that $\varphi_{0*}[f] = [\lambda * (\varphi_1 \circ f) * \lambda^{-1}]$ for all $[f] \in \pi_1(X, x_0)$ (thus φ_{0*} and φ_{1*} are conjugate).*
- (2) *If $\pi_1(Y, y_0)$ is abelian, then $\varphi_{0*} = \varphi_{1*}$.*

Proof. Part (1) follows immediately from Lemma 8.5: Since $\varphi_0(x_0) = y_0 = \varphi_1(x_0)$ and since λ is a closed path at y_0 , it follows that $[\lambda] \in \pi_1(Y, y_0)$ and

$$[\lambda * (\varphi_1 \circ f) * \lambda^{-1}] = [\lambda][\varphi_1 \circ f][\lambda^{-1}] = [\lambda]\varphi_{1*}[f][\lambda]^{-1}.$$

If $\pi_1(Y, y_0)$ is abelian, then

$$\varphi_{0*}[f] = [\lambda]\varphi_{1*}[f][\lambda]^{-1} = \varphi_{1*}[f].$$

$\square$

Theorem 8.7. *Let $\beta: X \rightarrow Y$ be a homotopy equivalence. Then $\beta_*: \pi_1(X, x_0) \rightarrow \pi_1(Y, \beta(x_0))$ is an isomorphism for every $x_0 \in X$.*

Proof. Since β is a homotopy equivalence, there is a continuous map $\alpha: Y \rightarrow X$ with $\alpha \circ \beta \simeq 1_X$ and $\beta \circ \alpha \simeq 1_Y$. Consider the following diagram:

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{\beta_*} & \pi_1(Y, \beta(x_0)) \\ \downarrow 1 & \searrow (\alpha \circ \beta)_* & \downarrow \alpha_* \\ \pi_1(X, x_0) & \xleftarrow{\psi} & \pi_1(X, \alpha\beta(x_0)). \end{array}$$

The left triangle commutes by Lemma 8.5. Since ψ is an isomorphism, also $(\alpha \circ \beta)_*$ is an isomorphism. Since π_1 is a functor, it follows that $(\alpha \circ \beta)_* = \alpha_* \circ \beta_*$. Thus

the right triangle commutes. Since $(\alpha \circ \beta)_*$ is a bijection, it now follows that α_* is a surjection and β_* is an injection.

Using the fact that $\beta \circ \alpha \simeq 1_Y$, it can be proved in the same way that α_* is injective and β_* is surjective. Therefore, β_* is an isomorphism. $\square$

Corollary 8.8. *Let X and Y be path-connected and assume they have the same homotopy type. Then, for every $x_0 \in X$ and for every $y_0 \in Y$,*

$$\pi_1(X, x_0) \cong \pi_1(Y, y_0).$$

Proof. Let $\beta: X \rightarrow Y$ be a homotopy equivalence. Theorem 8.7 implies that $\pi_1(X, x_0) \cong \pi_1(Y, \beta(x_0))$. By Theorem 8.3, the isomorphism classes of the fundamental groups of X and Y do not depend on the choice of basepoints. $\square$

Corollary 8.9. *Let X be a contractible space and let $x_0 \in X$. Then*

$$\pi_1(X, \{x_0\}) = \{0\}.$$

Proof. The claim follows immediately from Corollary 8.8 and the fact that the fundamental group of a one-point space is trivial. $\square$

Definition 8.10. A topological space X is called *simply connected*, if it is path-connected and if $\pi_1(X, x_0) = \{0\}$ for every $x_0 \in X$.

Notice that some authors do not require a simply connected space to be path-connected. They call a space simply connected if every path component of the space is simply connected in the sense of Definition 8.10.

Corollary 8.11. *Assume $\beta: (X, x_0) \rightarrow (Y, y_0)$ is (freely) nullhomotopic. Then the induced homomorphism $\beta_*: \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ is trivial.*

Proof. Let $k: X \rightarrow Y$, $x \mapsto y_1$, be a constant map. Then

$$k_*: \pi_1(X, x_0) \rightarrow \pi_1(Y, y_1), [f] \mapsto [k \circ f],$$

is trivial, since $k \circ f$ is a constant path at y_1 . Assume $\beta: (X, x_0) \mapsto (Y, y_0)$ is nullhomotopic, $\beta \simeq k$. By Lemma 8.5 there is an isomorphism ψ making the following diagram commute:

$$\begin{array}{ccc} \pi_1(X, x_0) & & \\ \beta_* \downarrow & \searrow k_* & \\ \pi_1(Y, y_0) & \xrightarrow[\psi]{} & \pi_1(Y, y_1) \end{array}$$

Then $\psi\beta_* = k_*$, and it follows that $\beta_* = \psi^{-1}k_*$ is trivial. $\square$

9. THE FUNDAMENTAL GROUP OF A CIRCLE

The exponential map $p: \mathbb{R} \rightarrow \mathbb{S}^1$ is defined by

$$p(t) = (\cos 2\pi t, \sin 2\pi t) = e^{i2\pi t}.$$

It is a continuous surjection and a group homomorphism $(\mathbb{R}, +) \rightarrow (\mathbb{S}^1, \cdot)$,

$$p(t + s) = e^{i2\pi(t+s)} = e^{i2\pi t} \cdot e^{i2\pi s} = p(t)p(s).$$

The kernel of p is $\ker p = \mathbb{Z}$. The restriction $p|: (-\frac{1}{2}, \frac{1}{2}) \rightarrow \mathbb{S}^1 \setminus \{-1\}$ is a homeomorphism. We denote the inverse function of $p|$ by $\log: \mathbb{S}^1 \setminus \{-1\} \rightarrow (-\frac{1}{2}, \frac{1}{2})$. For example, $p(0) = 1$ and thus $\log(1) = 0$.

Definition 9.1. Let X be a topological space and let $f: X \rightarrow \mathbb{S}^1$ be a function. A continuous function $f': X \rightarrow \mathbb{R}$ is called a *lift* of f , if $p \circ f' = f$.

Definition 9.2. Let $A \subset \mathbb{R}^n$ and $a \in A$. We say that A is *star-like at a* , if the line segment connecting x and a is in A for all $x \in A$.

Proposition 9.3. Let $X \subset \mathbb{R}^n$, $0 \in X$. Assume X is compact and star-like at 0. If $f: X \rightarrow \mathbb{S}^1$ is a continuous map and $t_0 \in \mathbb{R}$ satisfies $p(t_0) = f(0)$, then f has a unique lift $f': X \rightarrow \mathbb{R}$ satisfying $f'(0) = t_0$.

Proof. Since X is compact and $f: X \rightarrow \mathbb{S}^1$ is continuous, it follows that f is uniformly continuous. Thus there exists $\delta > 0$ such that $\|x - x'\| < \delta$ implies $\|f(x) - f(x')\| < 2$. Hence $f(x') \neq -f(x)$ if $\|x - x'\| < \delta$. Since X is bounded, there exists $n \in \mathbb{N}$ with $\frac{\|x\|}{n} < \delta$ for every $x \in X$. Then

$$\left\| \frac{(j+1)x}{n} - \frac{jx}{n} \right\| = \frac{\|x\|}{n} < \delta, \text{ for every } x \in X, \text{ and for every } j \in \{0, \dots, n-1\}.$$

Since X is star-like at 0, it follows that

$$\frac{jx}{n}, \frac{(j+1)x}{n} \in X, \text{ for every } x \in X, \text{ and for every } j \in \{0, \dots, n-1\}.$$

Thus

$$\left\| f\left(\frac{(j+1)x}{n}\right) - f\left(\frac{jx}{n}\right) \right\| < 2, \text{ for every } x \in X, \text{ and for every } j \in \{0, \dots, n-1\}.$$

It follows that the function

$$g_j: X \rightarrow \mathbb{S}^1 \setminus \{-1\}, \quad x \mapsto \frac{f\left(\frac{(j+1)x}{n}\right)}{f\left(\frac{jx}{n}\right)},$$

is well-defined and continuous for $0 \leq j \leq n-1$. Write f as a product of functions:

$$f(x) = f(0)g_0(x)g_1(x) \cdots g_{n-1}(x).$$

Let

$$f': X \rightarrow \mathbb{R}, \quad x \mapsto t_0 + \log(g_0(x)) + \cdots + \log(g_{n-1}(x)).$$

Then f' is continuous. It is a lift of f , since

$$\begin{aligned} p(f'(x)) &= p(t_0 + \log(g_0(x)) + \cdots + \log(g_{n-1}(x))) \\ &= p(t_0)p(\log(g_0(x))) \cdots p(\log(g_{n-1}(x))) \\ &= f(0)g_0(x) \cdots g_{n-1}(x) = f(x). \end{aligned}$$

Moreover,

$$\begin{aligned} f'(0) &= t_0 + \log(g_0(0)) + \cdots + \log(g_{n-1}(0)) \\ &= t_0 + \log(1) + \cdots + \log(1) \\ &= t_0, \end{aligned}$$

since $\log(1) = 0$.

It remains to prove the uniqueness: Assume $f', f'': X \rightarrow \mathbb{R}$ are lifts of f satisfying $f'(0) = f''(0) = t_0$. Let $h: X \rightarrow \mathbb{R}$, $x \mapsto f'(x) - f''(x)$. Then

$$\begin{aligned} p(h(x)) &= p(f'(x) - f''(x)) = \frac{p(f'(x))}{p(f''(x))} \quad (p \text{ is a homomorphism}) \\ &= \frac{f(x)}{f(x)} = 1, \quad \text{for all } x. \end{aligned}$$

Thus $h(X) \subset p^{-1}(1) = \mathbb{Z}$. Since X is star-like, it is path-connected. Therefore, $h(X)$ is a path-connected subset of $\mathbb{Z}$, which implies that it is a one-point set. Now, $h(0) = f'(0) - f''(0) = t_0 - t_0 = 0$. Thus $h(x) = 0$ for all $x \in X$, i.e., $f'(x) = f''(x)$ for all $x \in X$. $\square$

Notice that I and $I \times I$ are star-like at origin.

Corollary 9.4. (1) (*Uniqueness of path lifting*) Every path $\alpha: I \rightarrow \mathbb{S}^1$ with $\alpha(0) = 1$ has a unique lift $\alpha': I \rightarrow \mathbb{R}$ with $\alpha'(0) = 0$.

(2) (*Uniqueness for lifting homotopies*) Let $\alpha, \beta: I \rightarrow \mathbb{S}^1$ be paths with $\alpha(0) = \beta(0) = 1$. Let $\alpha', \beta': I \rightarrow \mathbb{R}$ be the lifts of α and β , respectively, with $\alpha'(0) = \beta'(0) = 0$. Every homotopy $F: \alpha \simeq \beta \text{ rel } \dot{I}$ has a unique lift $F': \alpha' \simeq \beta' \text{ rel } \dot{I}$.

Proof. Since $\alpha(0) = 1 = \beta(0)$, the first claim follows immediately from Proposition 9.3.

The homotopy F satisfies $F(0, 0) = 1$. Proposition 9.3 now implies that F has a unique lift $F': I^2 \rightarrow \mathbb{R}$ with $F'(0, 0) = 0$. The map $s \mapsto F'(s, 0)$ is a lift of the map $s \mapsto F(s, 0) = \alpha(s)$ and $F'(0, 0) = 0$. Uniqueness of the lift of α (part (1)) implies that $F'(s, 0) = \alpha'(s)$ for every $s \in I$.

The map $t \mapsto F'(0, t)$ is a lift of the map $t \mapsto F(0, t) = 1$. Thus it maps I to $p^{-1}(1) = \mathbb{Z}$, which implies that it is a constant map. Then $F'(0, 0) = 0$ implies that $F'(0, t) = 0$, for every $t \in I$. Similarly, $F'(s, 1) = \beta'(s)$ for every $s \in I$ and $F'(1, t)$ is constant. Thus $F': \alpha' \simeq \beta' \text{ rel } \dot{I}$. $\square$

Definition 9.5. Let $\alpha: I \rightarrow \mathbb{S}^1$ be a closed path with $\alpha(0) = \alpha(1) = 1$. Let $\alpha': I \rightarrow \mathbb{R}$ be the unique lift of α with $\alpha'(0) = 0$. We call $\alpha'(1) \in \mathbb{Z}$ the *degree* of α and denote it by $\deg \alpha$.

Notice that $p(\alpha'(1)) = \alpha(1) = 1$ implies that $\alpha'(1) \in \ker p = \mathbb{Z}$. If $\alpha \simeq \beta \text{ rel } \dot{I}$, then $\alpha' \simeq \beta' \text{ rel } \dot{I}$. Thus

$$\deg \alpha = \alpha'(1) = \beta'(1) = \deg \beta.$$

We obtain a function

$$\deg: \pi_1(\mathbb{S}^1, 1) \rightarrow \mathbb{Z}, \quad [\alpha] \mapsto \deg \alpha.$$

Proposition 9.6. *The function*

$$\deg: \pi_1(\mathbb{S}^1, 1) \rightarrow \mathbb{Z}, \quad [\alpha] \mapsto \deg \alpha,$$

is a group isomorphism.

Proof. The proof has three parts:

- (1) $\deg$ is an injection,
- (2) $\deg$ is a surjection,
- (3) $\deg$ is a group homomorphism.

We begin by proving part (1). Assume $\deg[\alpha] = \deg[\beta]$. Let α' and β' be the lifts of α and β , respectively. Then $\alpha'(1) = \beta'(1)$ and $\alpha'(0) = \beta'(0) = 0$. Let

$$F: I \times I \rightarrow \mathbb{R}, \quad (s, t) \mapsto (1 - t)\alpha'(s) + t\beta'(s).$$

Then $F: \alpha' \simeq \beta' \text{ rel } \dot{I}$, and hence $pF: \alpha \simeq \beta \text{ rel } \dot{I}$. Thus $[\alpha] = [\beta]$, and it follows that $\deg$ is an injection.

Let then $n \in \mathbb{Z}$. Let $\alpha_n: I \rightarrow \mathbb{S}^1$, $s \mapsto e^{i2\pi ns}$. The lift of α_n is $\alpha'_n: I \rightarrow \mathbb{R}$, $s \mapsto ns$. Then $\alpha'_n(1) = n$, which implies that $\deg[\alpha_n] = n$. It follows that $\deg$ is a surjection.

It remains to show that $\deg$ is a group homomorphism. By parts (1) and (2), $\pi_1(\mathbb{S}^1, 1) = \{[\alpha_n] \mid n \in \mathbb{Z}\}$. Since $\deg[\alpha_n] = n$, we will show that $\deg[\alpha_m * \alpha_n] = m + n$. Now,

$$(\alpha_m * \alpha_n)(s) = \begin{cases} e^{i2\pi ms}, & \text{if } 0 \leq s \leq \frac{1}{2}, \\ e^{i2\pi n(2s-1)}, & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases}$$

The path $\alpha_m * \alpha_n$ has a lift

$$(\alpha_m * \alpha_n)'(s) = \begin{cases} 2sm, & \text{if } 0 \leq s \leq \frac{1}{2}, \\ (2s-1)n + m, & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$

and $(\alpha_m * \alpha_n)'(1) = m + n$. Thus $\deg[\alpha_m * \alpha_n] = (\alpha_m * \alpha_n)'(1) = m + n$. $\square$

Corollary 9.7. $\mathbb{S}^1$ is not a retract of $\mathbb{D}^2$.

Proof. Counterassumption: Assume there is a retraction $r: \mathbb{D}^2 \rightarrow \mathbb{S}^1$. Let $i: \mathbb{S}^1 \hookrightarrow \mathbb{D}^2$ be the inclusion. Then both r and i are based maps with respect to the basepoint $1 \in \mathbb{S}^1$. Since $r \circ i = 1_{\mathbb{S}^1}$, it follows that

$$r_* \circ i_* = (r \circ i)_* = (1_{\mathbb{S}^1})_* = 1: \pi_1(\mathbb{S}^1, 1) \rightarrow \pi_1(\mathbb{S}^1, 1).$$

But

$$\mathbb{Z} \cong \pi_1(\mathbb{S}^1, 1) \xrightarrow{i_*} \pi_1(\mathbb{D}^2, 1) \xrightarrow{r_*} \pi_1(\mathbb{S}^1, 1) \cong \mathbb{Z}.$$

Since $\pi_1(\mathbb{D}^2, 1) = 0$, it follows that $r_* \circ i_*$ cannot be an isomorphism, a contradiction. $\square$

More generally we may conclude the following: Assume $r: X \rightarrow A$ is a based retraction, i.e., a based map that also is a retraction. Then

$$i_*: \pi_1(A, a_0) \rightarrow \pi_1(X, a_0) \text{ is an injection}$$

and

$$r_*: \pi_1(X, a_0) \rightarrow \pi_1(A, a_0) \text{ is a surjection.}$$

A *fixed point* of a map $f: X \rightarrow X$ is a point $x \in X$ such that $f(x) = x$.

Corollary 9.8. (Brouwer fixed point theorem) Every continuous map $f: \mathbb{D}^2 \rightarrow \mathbb{D}^2$ has a fixed point.

Proof. Counterassumption: There exists a continuous map $f: \mathbb{D}^2 \rightarrow \mathbb{D}^2$ such that $f(x) \neq x$, for every $x \in \mathbb{D}^2$. Define $g: \mathbb{D}^2 \rightarrow \mathbb{S}^1$ as follows: $g(x)$ is the point where the half line starting at $f(x)$ and going through x meets $\mathbb{S}^1$. Then g is continuous. Clearly, $g: \mathbb{D}^2 \rightarrow \mathbb{S}^1$ is a retraction, a contradiction. $\square$

Corollary 9.9. $\mathbb{S}^1$ is not simply connected. $\square$

Corollary 9.10. Closed paths f and g in $\mathbb{S}^1$ at 1 are homotopic rel $\dot{I}$, if and only if $\deg f = \deg g$.

Proof. Since $\deg: \pi_1(\mathbb{S}^1, 1) \rightarrow \mathbb{Z}$ is well-defined, it follows that $\deg f = \deg g$, if $f \simeq g \text{ rel } \dot{I}$. Since $\deg$ is injective, $\deg f = \deg g$ implies that $[f] = [g]$. $\square$

Theorem 9.11. (The Fundamental Theorem of Algebra) Every complex polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, where each $a_i \in \mathbb{C}$, $a_n \neq 0$ and $n > 0$, has a root $z_0 \in \mathbb{C}$.

Proof. If $a_n \neq 1$, then dividing p by a_n yields a polynomial that has the same roots as P . Therefore, we may assume that $a_n = 1$. Assume that $p(z) \neq 0$, for every $z \in \mathbb{C}$. Let

$$\mathcal{P}: \mathbb{S}^1 \times [0, \infty) \rightarrow \mathbb{S}^1, \quad (z, r) \mapsto \frac{p(rz)}{|p(rz)|} \cdot \frac{|p(r)|}{p(r)}.$$

Then $\mathcal{P}$ is continuous. Let

$$p_r: \mathbb{S}^1 \rightarrow \mathbb{S}^1, z \mapsto \mathcal{P}(z, r).$$

We will prove the following claims:

- (1) $[p_r] \in \pi_1(\mathbb{S}^1, 1)$ does not depend on r ,
- (2) $\deg p_0 = 0$,
- (3) $\deg p_1 = n$.

This will lead to a contradiction, since if $[p_0] = [p_1]$, then $\deg p_0 = \deg p_1$.

Proof of Claim (1): Let $r_1, r_2 \geq 0$. Let

$$F: \mathbb{S}^1 \times I \rightarrow \mathbb{S}^1, \quad (z, t) \mapsto \mathcal{P}(z, (1-t)r_1 + tr_2).$$

Then $F: p_{r_1} \simeq p_{r_2} \text{ rel } \{1\}$.

Proof of Claim (2): For every $z \in \mathbb{S}^1$,

$$p_0(z) = \frac{p(0 \cdot z)}{|p(0 \cdot z)|} \cdot \frac{|p(0)|}{p(0)} = 1.$$

Thus $[p_0] = 0$, which implies that $\deg P_0 = 0$.

Proof of Claim (3): Let $t > 0$. Then

$$p\left(\frac{z}{t}\right) = \frac{1}{t^n} (a_n z^n + a_{n-1} t z^{n-1} + \cdots + a_1 t^{n-1} z + a_0 t^n).$$

Thus

$$\frac{p(z/t)}{|p(z/t)|} = \frac{a_n z^n + a_{n-1} t z^{n-1} + \cdots + a_1 t^{n-1} z + a_0 t^n}{|a_n z^n + a_{n-1} t z^{n-1} + \cdots + a_1 t^{n-1} z + a_0 t^n|}.$$

As a function of (z, t) this can be extended to be continuous also at points $(z, 0)$, where the right side of the equation obtains the value

$$\frac{a_n z^n}{|a_n z^n|} = \frac{a_n}{|a_n|} \cdot \frac{1}{|z^n|} z^n = \frac{a_n}{|a_n|} z^n = z^n,$$

since $a_n = 1$. Thus the function

$$F: \mathbb{S}^1 \times [0, \infty) \rightarrow \mathbb{S}^1, \quad (z, t) \mapsto \begin{cases} z^n, & \text{if } t = 0, \\ \mathcal{P}(z, \frac{1}{t}), & \text{if } 0 < t \leq 1, \end{cases}$$

is continuous. Let $\mu_n: \mathbb{S}^1 \rightarrow \mathbb{S}^1$, $z \mapsto z^n$. Then:

- (1) $F(z, 0) = z^n = \mu_n(z)$, for all $z \in \mathbb{S}^1$.
- (2) $F(z, 1) = \mathcal{P}(z, 1) = p_1(z)$, for all $z \in \mathbb{S}^1$.
- (3)

$$F(1, t) = \begin{cases} 1^n = 1, & \text{if } t = 0, \\ \mathcal{P}(1, 1/t) = \frac{P(1 \cdot (1/t))}{|P(1 \cdot (1/t))|} \cdot \frac{|P(1/t)|}{P(1/t)} = 1, & \text{if } 0 < t \leq 1. \end{cases}$$

Thus $F: \mu_n \simeq p_1 \text{ rel } \{1\}$. Hence $\deg p_1 = \deg \mu_n$. Let

$$\beta: I \rightarrow \mathbb{S}^1, \quad t \mapsto e^{i2\pi t},$$

and let

$$\alpha_n: I \rightarrow \mathbb{S}^1, \quad t \mapsto (\mu_n \circ \beta)(t) = (e^{i2\pi t})^n = e^{in2\pi t}.$$

Then $\alpha_n(0) = \alpha_n(1) = 1$, and α_n has a lift

$$\alpha'_n: I \rightarrow \mathbb{R}, \quad t \mapsto nt.$$

Then $\alpha'_n(0) = 0$ and $\alpha'_n(1) = n$. Thus

$$\deg p_1 = \deg \mu_n = \deg \alpha_n = \alpha'_n(1) = n.$$

□

10. SEIFERT - VAN KAMPEN THEOREM

Recall the following:

Theorem 10.1. (*Lebesgue Number Theorem*) Let X be a compact metric space and let $\mathcal{U}$ be an open cover of X . Then there exists $\lambda > 0$ such that every open ball of radius less than λ lies in some element of $\mathcal{U}$.

Proof. See Chapter 3, Lemma 27.5 in [2].

□

Theorem 10.2. (*Seifert - van Kampen*) Let X be a topological space. Assume $X = X_1 \cup X_2$, where X_1 and X_2 are open, simply connected subsets of X . Assume $X_0 = X_1 \cap X_2 \neq \emptyset$ is path-connected. Then $\pi_1(X, x_0) = \{1\}$, for every $x_0 \in X$.

Proof. Since X_1 and X_2 are simply connected, they are path-connected. Since $X_1 \cap X_2$ is path-connected, it now follows that also X is path-connected. Thus, up to isomorphism, $\pi_1(X, x_0)$ does not depend on the choice of x_0 . Therefore, we may assume that $x_0 \in X_0$. Let $\alpha: I \rightarrow X$ be a closed path at x_0 . Then $\{\alpha^{-1}(X_1), \alpha^{-1}(X_2)\}$ is an open cover of I . Since I is compact, it follows from Theorem 10.1 that there exists $\lambda > 0$ such that every subinterval J of I , whose length is less than λ , lies in $\alpha^{-1}(X_1)$ or in $\alpha^{-1}(X_2)$.

Let $n \in \mathbb{N}$, where $\frac{1}{n} < \lambda$. Then

$$0 < \frac{1}{n} < \frac{2}{n} < \dots < \frac{n-1}{n} < 1,$$

and

$$\alpha([\frac{k-1}{n}, \frac{k}{n}]) \subset X_1 \text{ or } \alpha([\frac{k-1}{n}, \frac{k}{n}]) \subset X_2,$$

for every $k \in \{1, \dots, n\}$. By deleting some of the points $\frac{k}{n}$ if necessary, we obtain $t_i \in [0, 1]$:

$$0 = t_0 < t_1 < \dots < t_m = 1,$$

such that $\alpha([t_{i-1}, t_1])$ and $\alpha([t_i, t_{i+1}])$ lie in different sets X_1, X_2 . Then $\alpha(t_i) \in X_1 \cap X_2 = X_0$, for every i . Since X_0 is path-connected there is a path $\beta_i: I \rightarrow X_0$ from x_0 to $\alpha(t_i)$, for every i . Let $\alpha_i: I \rightarrow X$ be the path $\alpha|_{[t_{i-1}, t_i]}$ parametrized so that it is defined on the unit interval I . Then

$$\begin{aligned} \alpha &\simeq \alpha_1 * \alpha_2 * \dots * \alpha_m \\ &\simeq (\alpha_1 * \beta_1^{-1}) * (\beta_1 * \alpha_2 * \beta_2^{-1}) * \dots * (\beta_{m-1} * \alpha_m) \text{ rel } \dot{I}. \end{aligned}$$

Now, each of the paths $\alpha_1 * \beta_1^{-1}, \dots, \beta_{m-1} * \alpha_m$ in the concatenation above is a closed path at x_0 , and each of these paths lies in X_1 or in X_2 . Since X_1 and X_2 are simply connected, each of these paths is nullhomotopic rel $\dot{I}$. Thus also α is nullhomotopic rel $\dot{I}$. $\square$

Lemma 10.3. *There is a homeomorphism $\mathbb{S}^n \setminus \{e_{n+1}\} \rightarrow \mathbb{R}^n$, for every $n \geq 1$, where $e_{n+1} = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$. Also, there is a homeomorphism $\mathbb{D}^n / \mathbb{S}^{n-1} \rightarrow \mathbb{S}^n$.*

Proof. Let

$$\pi: \mathbb{D}^n \rightarrow \mathbb{S}^n, \quad y \mapsto (2\sqrt{1 - \|y\|^2} \, y, 2\|y\|^2 - 1).$$

Since $\|\pi(y)\| = 1$, for all $y \in \mathbb{D}^n$, it follows that π is well-defined. Also, $\pi^{-1}(\{e_{n+1}\}) = \mathbb{S}^{n-1}$. The restriction of π to $\mathring{\mathbb{D}}^n$ is a homeomorphism $\mathring{\mathbb{D}}^n \rightarrow \mathbb{S}^n \setminus \{e_{n+1}\}$. The inverse is given by

$$\varrho: \mathbb{S}^n \setminus \{e_{n+1}\} \rightarrow \mathring{\mathbb{D}}^n, \quad (z, t) \mapsto \frac{z}{\sqrt{2(1-t)}}.$$

The map π induces a continuous bijection

$$\bar{\pi}: \mathbb{D}^n / \mathbb{S}^{n-1} \rightarrow \mathbb{S}^n,$$

so that $\pi = \bar{\pi} \circ p$, where $p: \mathbb{D}^n \rightarrow \mathbb{D}^n / \mathbb{S}^{n-1}$ is the quotient map. Since $\mathbb{D}^n$ is compact and p is a continuous surjection, it follows that $\mathbb{D}^n / \mathbb{S}^{n-1}$ is compact. Since $\mathbb{S}^n$ is Hausdorff, it follows that $\bar{\pi}$ is a closed map. Hence $\bar{\pi}$ is a homeomorphism. The map

$$\pi': \mathring{\mathbb{D}}^n \rightarrow \mathbb{R}^n, \quad y \mapsto \frac{y}{1 - \|y\|},$$

is a homeomorphism with the inverse

$$\varrho': \mathbb{R}^n \rightarrow \mathring{\mathbb{D}}^n, \quad z \mapsto \frac{z}{1 + \|z\|}.$$

Then the composed map

$$\pi' \circ \varrho: \mathbb{S}^n \setminus \{e_{n+1}\} \rightarrow \mathbb{R}^n$$

is a homeomorphism. $\square$

Notice that in the previous lemma the point e_{n+1} could be replaced by any $x_0 \in \mathbb{S}^n$.

Theorem 10.4. *The sphere $\mathbb{S}^n$ is simply connected for all $n \geq 2$.*

Proof. Let

$$U_1 = \mathbb{S}^n \setminus \{e_{n+1}\}, \text{ and } U_2 = \mathbb{S}^n \setminus \{-e_{n+1}\}.$$

Then $\mathbb{S}^n = U_1 \cup U_2$. By Lemma 10.3, both U_1 and U_2 are homeomorphic to $\mathbb{R}^n$. Now, $\mathbb{R}^n$ is path-connected for all n , and $U_1 \cap U_2 \neq \emptyset$ for $n \geq 1$. Thus $\mathbb{S}^n$ is path-connected for $n \geq 1$. Since $\mathbb{R}^n$ is simply connected, it follows that also U_1 and U_2 are simply connected. Also,

$$f: U_1 \cap U_2 \rightarrow \mathbb{S}^{n-1} \times (-1, 1), \quad (z, t) \mapsto \left(\frac{z}{\|z\|}, t \right),$$

is a homeomorphism. The inverse of f is

$$f^{-1}: \mathbb{S}^{n-1} \times (-1, 1) \rightarrow U_1 \cap U_2, \quad (z, t) \mapsto (\sqrt{1-t^2}z, t).$$

Thus $U_1 \cap U_2$ is path-connected for $n \geq 2$. It follows from the Seifert-van Kampen theorem that $\pi_1(\mathbb{S}^n, x_0) = \{1\}$, for all $x_0 \in U_1 \cap U_2 = \mathbb{S}^n \setminus \{\pm e_{n+1}\}$. Since $\mathbb{S}^n$ is path-connected, it follows from Theorem 8.3 that also $\pi_1(\mathbb{S}^n, e_{n+1}) = \{1\} = \pi_1(\mathbb{S}^n, -e_{n+1})$, for $n \geq 2$. $\square$

Corollary 10.5. *The spheres $\mathbb{S}^1$ and $\mathbb{S}^n$ do not have the same homotopy type for $n > 1$.*

Proof. By Corollary 8.8, topological spaces that have the same homotopy type, have isomorphic fundamental groups. $\square$

Corollary 10.6. *The euclidean spaces $\mathbb{R}^2$ and $\mathbb{R}^n$ are not homeomorphic for $n \geq 3$.*

Proof. Let's make a counterassumption, and assume there is a homeomorphism $f: \mathbb{R}^2 \rightarrow \mathbb{R}^n$. If $f(0) = x_0$, then

$$f': \mathbb{R}^2 \rightarrow \mathbb{R}^n, \quad x \mapsto f(x) - x_0,$$

is a homeomorphism satisfying $f'(0) = 0$. Let $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$. Since f' is a bijection, it follows that $f'(e_1) \neq 0$. Let $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear bijection with $A(f'(e_1)) = e_1$. Then

$$g = A \circ f': \mathbb{R}^2 \rightarrow \mathbb{R}^n$$

is a homeomorphism,

$$g(0) = (A \circ f')(0) = A(f'(0)) = A(0) = 0, \text{ and } g(e_1) = A(f'(e_1)) = e_1.$$

The restriction of $A \circ f'$,

$$h: \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}^n \setminus \{0\}, \quad x \mapsto g(x),$$

is a homeomorphism and $h(e_1) = e_1$. Let $i: \mathbb{S}^1 \hookrightarrow \mathbb{R}^2 \setminus \{0\}$ be the inclusion and let

$$r: \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{S}^{n-1}, \quad x \mapsto \frac{x}{\|x\|}.$$

Then i and r are homotopy equivalences. Thus the composed map

$$\mathbb{S}^1 \xrightarrow{i} \mathbb{R}^2 \setminus \{0\} \xrightarrow{h} \mathbb{R}^n \setminus \{0\} \xrightarrow{r} \mathbb{S}^{n-1}$$

is a homotopy equivalence. This is a contradiction. $\square$

11. TOPOLOGICAL GROUPS AND H -SPACES

Definition 11.1. A *topological group* G is a group equipped with a topology such that sets consisting of one point are closed and

- (1) the multiplication map $\mu: G \times G \rightarrow G$, $(x, y) \mapsto xy$, is continuous if $G \times G$ has the product topology,
- (2) the inversion map $i: G \rightarrow G$, $x \mapsto x^{-1}$, is continuous.

Example 11.2. The groups $(\mathbb{R}, +)$ and $(\mathbb{Z}, +)$ are topological groups. The circle $\mathbb{S}^1$ equipped with complex multiplication is a topological group. Any group equipped with the discrete topology is a topological group.

Lemma 11.3. *Topological groups are Hausdorff spaces.*

Proof. Recall that a topological space X is Hausdorff if and only if its diagonal $\Delta X = \{(x, x) \mid x \in X\}$ is closed in $X \times X$. Let G be a topological group. The map $f: G \times G \rightarrow G$, $(g, h) \mapsto gh^{-1}$ is continuous. Let e be the identity element of G . Since $\{e\}$ is closed in G , it follows that $\Delta G = f^{-1}(e)$ is closed in $G \times G$. Thus G is Hausdorff. $\square$

Let G be a topological group, and let $h \in G$. Then h can be considered as a homeomorphism $G \rightarrow G$, $g \mapsto hg$. Then h is continuous as a composition of continuous maps $(c_h, 1_G): G \rightarrow G \times G$, $g \mapsto (h, g)$, and $\mu: G \times G \rightarrow G$. Similarly, the inverse h^{-1} is continuous.

Let H be a closed normal subgroup of a topological group G . Then G/H is a group. Let $p: G \rightarrow G/H$, $g \mapsto gH$. Then p is a surjection. Equip G/H with the quotient topology from G : a subset U of G/H is open in G/H if and only if $p^{-1}(U)$ is open in G . Then p is a continuous function.

Lemma 11.4. *The quotient map $p: G \rightarrow G/H$ is an open map.*

Proof. Let U be open in G . Let $g \in G$. Then the set $U_g = \{xg \mid x \in U\}$ is open in U , since $g: G \rightarrow G$, $x \mapsto xg$, is a homeomorphism. Now, $p^{-1}(p(U)) = \bigcup_{h \in H} Uh$ is open in G as a union of open sets. Thus $p(U)$ is open in G/H . $\square$

Proposition 11.5. *Let G be a topological group and let H be a closed, normal subgroup of G . Then G/H is a topological group.*

Proof. Homework. $\square$

Proposition 11.6. *Let G be a topological group and let H be an open subgroup of G . Then H is closed in G .*

Proof. Homework. □

Definition 11.7. Let (X, x_0) be a pointed space. Assume there is a pointed map

$$m: (X \times X, (x_0, x_0)) \rightarrow (X, x_0)$$

such that the pointed maps $m(, x_0)$ and $m(x_0,)$ are homotopic to $1_X \text{ rel } \{x_0\}$. Then (X, x_0) is called an *H-space* (after H. Hopf).

Example 11.8. Every topological group X with identity x_0 is an *H-space*.

Let (X, x_0) be an *H-space*. Let $k: X \rightarrow X, x \mapsto x_0$. Then $m(x_0,) = m \circ (k, 1_X)$, where $(k, 1_X): X \rightarrow X \times X, x \mapsto (x_0, x)$. Similarly, $m(, x_0) = m \circ (1_X, k)$. Thus both $m \circ (k, 1_X)$ and $m \circ (1_X, k)$ are homotopic to $1_X \text{ rel } \{x_0\}$.

Recall the following: Let G and H be groups with identity elements e and e' , respectively. Let $x \in G$ and let $y \in H$. Then

$$(x, e')(e, y) = (x, y) = (e, y)(x, e').$$

Theorem 11.9. Let (X, x_0) be an *H-space*. Then $\pi_1(X, x_0)$ is abelian.

Proof. By Theorem 8.4, the map

$$\theta: \pi_1(X, x_0) \times \pi_1(X, x_0) \rightarrow \pi_1(X \times X, (x_0, x_0)), ([f], [g]) \mapsto [(f, g)],$$

is a group isomorphism. (Here $(f, g): I \rightarrow X \times X, t \mapsto (f(t), g(t))$.) Let $[f], [g] \in \pi_1(X, x_0)$. Then

$$\begin{aligned} [g] &= (m \circ (k, 1_X))_* [g] \quad (\text{definition of } H\text{-space}) \\ &= m_*(k, 1_X)_* [g] \\ &= m_*[(k, 1_X) \circ g] \\ &= m_*[(kg, g)] \\ &= m_*\theta([kg], [g]) \quad (\text{definition of } G) \\ &= m_*\theta(e, [g]), \end{aligned}$$

where $e = [k]$ is the identity element of $\pi_1(X, x_0)$. Similarly, $[f] = m_*\theta([f], e)$, since $m \circ (1_X, k) \simeq 1_X \text{ rel } \{x_0\}$. The composition

$$m_*\theta: \pi_1(X, x_0) \times \pi_1(X, x_0) \xrightarrow{\theta} \pi_1(X \times X, (x_0, x_0)) \xrightarrow{m_*} \pi_1(X, x_0)$$

is a homomorphism. Therefore,

$$\begin{aligned} m_*\theta([f], [g]) &= m_*\theta((e, [g])([f], e)) \\ &= m_*\theta((e, [g]))m_*\theta([f], e) \\ &= [g][f]. \end{aligned}$$

By writing

$$([f], [g]) = ([f], e)(e, [g]),$$

one sees that

$$m_*\theta([f], [g]) = [f][g].$$

Thus $[g][f] = [f][g]$ and it follows that $\pi_1(X, x_0)$ is abelian. □

Corollary 11.10. Let G be a topological group. Then $\pi_1(G, e)$ is abelian. □

12. EILENBERG - STEENROD AXIOMS

Let X be a topological space and let $A \subset X$. Then (X, A) is called a *topological pair*. A continuous function $f: (X, A) \rightarrow (Y, B)$ means a continuous function $f: X \rightarrow Y$ with $f(A) \subset B$. We write X for $(X, \emptyset)$.

A *homology theory* H defined for all topological pairs (X, A) and for all continuous functions $f: (X, A) \rightarrow (Y, B)$ consists of the following: For every topological pair (X, A) and for every $n \in \mathbb{N} \cup \{0\}$, there is an abelian group $H_n(X, A)$. Every continuous function $f: (X, A) \rightarrow (Y, B)$ induces a group homomorphism

$$f_*: H_n(X, A) \rightarrow H_n(Y, B), \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

For every (X, A) and for every $n \in \mathbb{N} \cup \{0\}$, there is a group homomorphism

$$\Delta: H_n(X, A) \rightarrow H_{n-1}(A),$$

such that the following axioms (called *Eilenberg - Steenrod axioms*) hold:

A1. The identity map $\text{id}: (X, A) \rightarrow (X, A)$ induces

$$\text{id}_* = \text{id}: H_n(X, A) \rightarrow H_n(X, A),$$

for all $n \geq 0$.

A2. If $f: (X, A) \rightarrow (Y, B)$ and $g: (Y, B) \rightarrow (Z, C)$ are continuous, then

$$(g \circ f)_* = g_* \circ f_*: H_n(X, A) \rightarrow H_n(Z, C),$$

for all $n \geq 0$.

A3. If $f: (X, A) \rightarrow (Y, B)$ is continuous, then the diagram

$$\begin{array}{ccc} H_n(X, A) & \xrightarrow{\Delta} & H_{n-1}(A) \\ f_* \downarrow & & \downarrow (f|A)_* \\ H_n(Y, B) & \xrightarrow{\Delta} & H_{n-1}(B). \end{array}$$

commutes for all $n \geq 1$.

A4. (*Exactness axiom*) If (X, A) is a topological space, and if $i: A \rightarrow X$ and $j: (X, \emptyset) \rightarrow (X, A)$ are inclusions, then the sequence

$$\cdots \xrightarrow{j_*} H_{n+1}(X, A) \xrightarrow{\Delta} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\Delta} H_{n-1}(A) \xrightarrow{i_*} \cdots$$

is exact.

(A sequence of abelian groups and homomorphisms

$$\cdots \longrightarrow S_{n+1} \xrightarrow{f_{n+1}} S_n \xrightarrow{f_n} S_{n-1} \cdots$$

is called *exact* if $\text{im} f_{n+1} = \ker f_n$ for all n .)

A5. (*Homotopy axiom*) If $f, g: (X, A) \rightarrow (Y, B)$ are homotopic (i.e., there is a homotopy $F: (X \times I, A \times I) \rightarrow (Y, B)$ with $F_0 = f$ and $F_1 = g$), then $f_* = g_*: H_n(X, A) \rightarrow H_n(Y, B)$ for all $n \geq 0$.

A6. (*Excision axiom*) For every topological pair (X, A) and for every open subset U of X with $\overline{U} \subset A$, the inclusion $i: (X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces group isomorphisms

$$i_*: H_n(X \setminus U, A \setminus U) \rightarrow H_n(X, A),$$

for all $n \geq 0$.

A7. (*Dimension axiom*) If X is a one-point space, then $H_n(X) = 0$ for all $n > 0$. (The group $H_0(X, \emptyset)$ is called the *coefficient group*.)

Notice that it varies how the Eilenberg-Steenrod axioms are presented in literature. Sometimes axioms 1 and 2, although assumed to hold, are not considered as axioms. Sometimes an extra axiom like the *additivity axiom* or the *axiom of compact supports* is added. Of all these axioms the dimension axiom may seem the least interesting. However, almost the opposite is true. There are mathematical theories that resemble homology theory, for example cobordism theory and K-theory. These theories satisfy all the Eilenberg-Steenrod axioms except the dimension axiom. Such theories are called *extraordinary* or *generalized homology theories*.

13. SINGULAR HOMOLOGY THEORY

Theorem 13.1. *There exists a homology theory H defined for all topological pairs (X, A) and for all continuous maps $f: (X, A) \rightarrow (Y, B)$ satisfying all 7 Eilenberg-Steenrod axioms.*

We will prove Theorem 13.1 by constructing the *singular homology theory* (by S. Eilenberg, around 1947).

Recall the the standard n -simplex, $n \geq 0$, is

$$\Delta^n = \{(t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n t_i = 1, t_i \geq 0, \text{ for all } i\}.$$

Then, for example,

$$\Delta^0 = \{t_0 \in \mathbb{R} \mid t_0 = 1\} = \{1\}$$

and

$$\Delta^1 = \{(t_0, t_1) \in \mathbb{R}^2 \mid t_0 + t_1 = 1, t_0, t_1 \geq 0\}.$$

Thus, if $p_i = (0, \dots, 0, 1, 0, \dots, 0)$, then Δ^n is the convex hull of $p_0, \dots, p_n$. Remember, that here the indexing goes from 0 to n , the coordinate 1 of p_i corresponds to the index i . For every $n \geq 1$ and $0 \leq j \leq n$, define

$$e_n^j = e^j: \Delta^{n-1} \rightarrow \Delta^n, \quad (t_0, \dots, t_{n-1}) \mapsto (t_0, \dots, t_{j-1}, 0, t_j, \dots, t_{n-1}).$$

Thus e_n^j adds an extra coordinate 0 between t_{j-1} and t_j . For $n = 1$, there are the maps

$$e^0: \Delta^0 \rightarrow \Delta^1, \quad 1 \mapsto (0, 1), \quad \text{and} \quad e^1: \Delta^0 \rightarrow \Delta^1, \quad 1 \mapsto (1, 0).$$

For $n = 2$, there are three maps

$$\begin{aligned} e^0: \Delta^1 &\rightarrow \Delta^2, (t_0, t_1) \mapsto (0, t_0, t_1), \\ e^1: \Delta^1 &\rightarrow \Delta^2, (t_0, t_1) \mapsto (t_0, 0, t_1), \\ e^2: \Delta^1 &\rightarrow \Delta^2, (t_0, t_1) \mapsto (t_0, t_1, 0). \end{aligned}$$

Lemma 13.2. *If $2 \leq n$ and $0 \leq k < j \leq n$, then*

$$e_n^j \circ e_{n-1}^k = e_n^k \circ e_{n-1}^{j-1}: \Delta^{n-2} \rightarrow \Delta^n.$$

Proof. Assume first $k = j - 1$. Then

$$\begin{aligned} e_n^j \circ e_{n-1}^k(t_0, \dots, t_{n-2}) &= e_n^j(t_0, \dots, t_{k-1}, 0, t_k, \dots, t_{n-2}) \\ &= (t_0, \dots, t_{k-1}, 0, 0, t_k, \dots, t_{n-2}) \end{aligned}$$

and

$$\begin{aligned} e_n^k \circ e_{n-1}^{j-1}(t_0, \dots, t_{n-2}) &= e_n^k(t_0, \dots, t_{j-2}, 0, t_{j-1}, \dots, t_{n-2}) \\ &= (t_0, \dots, t_{j-2}, 0, 0, t_{j-1}, \dots, t_{n-2}). \end{aligned}$$

Since $j - 2 = k - 1$, it follows that $e_n^j \circ e_{n-1}^k(t_0, \dots, t_{n-2}) = e_n^k \circ e_{n-1}^{j-1}(t_0, \dots, t_{n-2})$, for every $(t_0, \dots, t_{n-2}) \in \Delta^{n-2}$. Therefore, $e_n^j \circ e_{n-1}^k = e_n^k \circ e_{n-1}^{j-1}$.

The case $k < j - 1$ is left as an exercise. $\square$

The map e_n^j is called the j^{th} *face map* of the simplex Δ^n .

Definition 13.3. Let X be a topological space. A continuous map $T: \Delta^n \rightarrow X$ is called a *singular n -simplex* of X .

Define

$$\text{Map}(\Delta^n; X)$$

to be the set of all continuous functions $\Delta^n \rightarrow X$, i.e., the set of all singular n -simplices of X . Let $T \in \text{Map}(\Delta^n; X)$. Let $\mathbb{Z}_T \cong \mathbb{Z}$ be the infinite cyclic group generated by T . Thus the elements of $\mathbb{Z}_T$ are of the form mT , where $m \in \mathbb{Z}$. Define

$$S_n(X) = \sum_{T \in \text{Map}(\Delta^n; X)} \oplus \mathbb{Z}_T, \quad n \geq 0,$$

to be the free abelian group with basis all singular n -simplices in X . Therefore, the elements of $S_n(X)$ are of the form

$$\sum_{T \in \text{Map}(\Delta^n; X)} n_T T,$$

where $n_T \in \mathbb{Z}$, and $n_T = 0$ except for finitely many T (i.e., of the form $\sum_{i=1}^k n_i T_i = n_1 T_1 + \dots + n_k T_k$). The elements of $S_n(X)$ are called (*singular*) n -chains in X .

Let

$$T: \Delta^n \rightarrow X, (t_0, \dots, t_n) \mapsto T(t_0, \dots, t_n).$$

The j -face of T , $0 \leq j \leq n$, is

$$T^{(j)} = T \circ e^j: \Delta^{n-1} \rightarrow X, (t_0, \dots, t_{n-1}) \mapsto T(t_0, \dots, t_{j-1}, 0, t_j, \dots, t_{n-1}).$$

Define $S_{-1}(X)$ to be $\{0\}$ (sometimes it is convenient to define $S_n(X) = \{0\}$, for all negative integers n). Define the *boundary homomorphism*

$$\partial_n: S_n(X) \rightarrow S_{n-1}(X)$$

as follows: If $T \in \text{Map}(\Delta^n; X)$, we define

$$\partial_n T = \sum_{i=0}^n (-1)^i T^{(i)},$$

and $\partial_n: S_n(X) \rightarrow S_{n-1}(X)$ is obtained by requiring ∂_n to be *linear*, for $n > 0$. This means that

$$\partial_n(n_1 T_1 + \cdots + n_k T_k) = n_1 \partial_n T_1 + \cdots + n_k \partial_n T_k.$$

If $n = 0$, then $\partial_0: S_0(X) \rightarrow \{0\}$ is the constant map 0.

We have constructed a sequence of free abelian groups and homomorphisms

$$\cdots \xrightarrow{\partial_{n+1}} S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\partial_0} 0,$$

called the *singular chain complex* of X and denoted by

$$(S_*(X), \partial) \text{ or by } S_*(X).$$

Proposition 13.4. *The composed homomorphism*

$$S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \xrightarrow{\partial_{n-1}} S_{n-2}(X)$$

is a zero homomorphism, i.e., $\partial_{n-1} \circ \partial_n = 0$.

Proof. Let $T: \Delta^n \rightarrow X$ be a singular n -simplex of X . Then

$$\begin{aligned} \partial_{n-1} \partial_n(T) &= \partial_{n-1} \left(\sum_{j=0}^n (-1)^j T^{(j)} \right) = \sum_{j=0}^n (-1)^j \partial_{n-1} T^{(j)} \\ &= \sum_{j=0}^n (-1)^j \left(\sum_{k=0}^{n-1} (-1)^k (T^{(j)})^{(k)} \right) \\ &= \sum_{0 \leq k < j \leq n} (-1)^{j+k} (T^{(j)})^{(k)} + \sum_{0 \leq j \leq k \leq n-1} (-1)^{j+k} (T^{(j)})^{(k)}. \quad (*) \end{aligned}$$

By Lemma 13.2, $(T^{(j)})^{(k)} = (T^{(k)})^{(j-1)}$, when $0 \leq k < j \leq n$. Thus the first sum in $(*)$ equals

$$\begin{aligned} \sum_{0 \leq k < j \leq n} (-1)^{j+k} (T^{(k)})^{(j-1)} &= \sum_{0 \leq k \leq j-1 \leq n-1} -(-1)^{(j-1)+k} (T^{(k)})^{(j-1)} \\ &= -(\text{the second sum in } (*)). \end{aligned}$$

(Switch here: $j \leftrightarrow k$, $k \leftrightarrow j-1$.) Thus $\partial_{n-1} \partial_n(T) = 0$. Since $\partial_{n-1} \partial_n(T) = 0$ for all $T \in \text{map}(\Delta^n, X)$, it follows that

$$\partial_{n-1} \partial_n = 0: S_n(X) \rightarrow S_{n-2}(X).$$

□

Definition 13.5. Let $c \in S_n(X)$.

- (1) If $\partial_n c = 0$, we call c a (*singular*) n -cycle.
- (2) If there is such $d \in S_{n+1}(X)$ that $\partial_{n+1} d = c$, we call c a (*singular*) n -boundary.

Denote

$$Z_n(X) = \ker \partial_n \text{ and } B_n(X) = \operatorname{im} \partial_{n+1}.$$

Then $Z_n(X)$ and $B_n(X)$ are subgroups of $S_n(X)$. By Proposition 13.4, $B_n(X) \subset Z_n(X)$, for every n .

Definition 13.6. For each $n \geq 0$, the n^{th} (*singular*) *homology group* of a space X is

$$H_n(X) = Z_n(X)/B_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}.$$

Let X and Y be topological spaces and let $f: X \rightarrow Y$ be continuous. Let $T: \Delta^n \rightarrow X$ be a singular n -simplex of X . Then $f \circ T: \Delta^n \rightarrow Y$ is a singular n -simplex of Y . Define a homomorphism

$$f_{\#}: S_n(X) \rightarrow S_n(Y)$$

by setting $f_{\#}(T) = f \circ T \in S_n(Y)$, for all $T \in \operatorname{Map}(\Delta^n; X)$ and extending by linearity,

$$f_{\#}\left(\sum_{i=1}^k n_i T_i\right) = \sum_{i=1}^k n_i (f \circ T_i),$$

where $n_i \in \mathbb{Z}$, for every $i \in \{1, \dots, k\}$.

Lemma 13.7. If $f: X \rightarrow Y$ is continuous, then $\partial_n f_{\#} = f_{\#} \partial_n$, i.e., for every $n \geq 0$, the diagram

$$\begin{array}{ccc} S_n(X) & \xrightarrow{\partial_n} & S_{n-1}(X) \\ f_{\#} \downarrow & & \downarrow f_{\#} \\ S_n(Y) & \xrightarrow{\partial_n} & S_{n-1}(Y). \end{array}$$

commutes.

Proof. Let $T \in \operatorname{Map}(\Delta^n; X)$. Then

$$\begin{aligned} (\partial_n \circ f_{\#})(T) &= \partial_n(f_{\#}(T)) = \partial_n(f \circ T) = \sum_{j=0}^n (-1)^j (f \circ T \circ e^j) \\ &= f_{\#}\left(\sum_{j=0}^n (-1)^j (T \circ e^j)\right) = f_{\#}(\partial_n T) = (f_{\#} \circ \partial_n)(T). \end{aligned}$$

Since the singular n -simplices T generate $S_n(X)$ and since $f_{\#}$ and ∂_n are homomorphisms, it follows that $\partial_n \circ f_{\#} = f_{\#} \circ \partial_n$. $\square$

Lemma 13.8. Let $f: X \rightarrow Y$ be continuous. Then for every $n \geq 0$,

- (1) $f_{\#}(Z_n(X)) \subset Z_n(Y)$, and
- (2) $f_{\#}(B_n(X)) \subset B_n(Y)$.

Proof. (1):

$$\begin{aligned} c \in Z_n(X) &\Rightarrow \partial_n(c) = 0 \Rightarrow \partial_n(f_{\#}(c)) = (\partial_n \circ f_{\#})(c) \stackrel{13.7}{=} (f_{\#} \circ \partial_n)(c) \\ &= f_{\#}(\partial_n(c)) = f_{\#}(0) = 0 \Rightarrow f_{\#}(c) \in Z_n(Y). \end{aligned}$$

(2):

$$\begin{aligned} c \in B_n(X) &\Rightarrow \exists d \in S_{n+1}(X): C = \partial_{n+1}(d) \\ &\Rightarrow f_{\#}(c) = f_{\#}(\partial_{n+1}(d)) = (f_{\#} \circ \partial_{n+1})(d) \stackrel{13.7}{=} (\partial_{n+1} \circ f_{\#})(d) \\ &= \partial_{n+1}(f_{\#}(d)) \Rightarrow f_{\#}(c) \in B_n(Y). \end{aligned}$$

□

It follows from Lemma 13.8, that $f_{\#}: S_n(X) \rightarrow S_n(Y)$ induces a homomorphism

$$f_*: Z_n(X)/B_n(X) \rightarrow Z_n(Y)/B_n(Y), \quad cB_n(X) \mapsto f_{\#}(c)B_n(Y),$$

i.e., a homomorphism

$$f_*: H_n(X) \rightarrow H_n(Y).$$

If $c \in Z_n(X)$, we write

$$[c] = cB_n(X) \in H_n(X),$$

and call $[c]$ the *homology class* of c . Since $Z_n(X)$ is an abelian group, we may sometimes write $c + B_n(X)$ instead of $cB_n(X)$. If $c, c' \in Z_n(X)$, then

$$\begin{aligned} [c] = [c'] \in H_n(X) &\iff cB_n(X) = c'B_n(X) \\ &\iff c = c' + b, \text{ for some } b \in B_n(X). \end{aligned}$$

Theorem 13.9. *For every $n \geq 0$, $H_n: \text{Top} \rightarrow \text{Ab}$ is a functor.*

Proof. For every continuous $f: X \rightarrow Y$, define

$$H_n(f) = f_*: H_n(X) \rightarrow H_n(Y), \quad cB_n(X) \mapsto f_{\#}(c)B_n(Y).$$

If $f: X \rightarrow X$ is the identity function id_X , then $f_{\#} = \text{id}: S_n(X) \rightarrow S_n(X)$, for all $n \geq 0$. Thus $H_n(f): H_n(X) \rightarrow H_n(Y)$ is the identity homomorphism, for all $n \geq 0$.

Let then $f: X \rightarrow Y$ and $g: Y \rightarrow W$ be continuous. Then $g \circ f: X \rightarrow W$ is continuous, and

$$(g \circ f)_{\#} = g_{\#} \circ f_{\#}: S_n(X) \rightarrow S_n(W),$$

for all $n \geq 0$: For any continuous function $T: \Delta^n \rightarrow X$,

$$g_{\#} \circ f_{\#}: S_n(X) \rightarrow S_n(Y) \rightarrow S_n(W), \quad T \mapsto f \circ T \mapsto g \circ (f \circ T),$$

and

$$(g \circ f)_{\#}: S_n(X) \rightarrow S_n(W), \quad T \mapsto (g \circ f)(T),$$

where $g \circ (f \circ T) = (g \circ f)(T)$. Thus

$$g_* \circ f_* = (g \circ f)_*: H_n(X) \rightarrow H_n(W),$$

i.e.,

$$H_n(g) \circ H_n(f) = H_n(g \circ f): H_n(X) \rightarrow H_n(W),$$

for all $n \geq 0$. $\square$

Proposition 13.10. *If topological spaces X and Y are homeomorphic, then $H_n(X)$ and $H_n(Y)$ are isomorphic, for all $n \geq 0$.*

Proof. Let $f: X \rightarrow Y$ be a homeomorphism. Then there is a continuous function $g: Y \rightarrow X$ with $g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$. Let $n \geq 0$. The functions f and g induce homomorphisms

$$f_*: H_n(X) \rightarrow H_n(Y) \quad \text{and} \quad g_*: H_n(Y) \rightarrow H_n(X),$$

respectively. By Theorem 13.9,

$$g_* \circ f_* = (g \circ f)_* = (\text{id}_X)_* = \text{id}: H_n(X) \rightarrow H_n(X),$$

and

$$f_* \circ g_* = (f \circ g)_* = (\text{id}_Y)_* = \text{id}: H_n(Y) \rightarrow H_n(Y).$$

Thus f_* is an isomorphism with the inverse $f_*^{-1} = g_*$. $\square$

14. DIMENSION AXIOM AND EXAMPLES

Example 14.1. Let $X = \{p\}$ be a one-point space. For every $n \geq 0$, there is exactly one function $T_n: \Delta^n \rightarrow \{p\}$. Thus $S_n(X) = \mathbb{Z}_{T_n}$. The singular chain complex of X is

$$\cdots \longrightarrow S_{n+1}(\{p\}) \xrightarrow{\partial_{n+1}} S_n(\{p\}) \xrightarrow{\partial_n} S_{n-1}(\{p\}) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} S_0(\{p\}) \longrightarrow 0,$$

which then equals

$$\cdots \longrightarrow \mathbb{Z}_{T_{n+1}} \xrightarrow{\partial_{n+1}} \mathbb{Z}_{T_n} \xrightarrow{\partial_n} \mathbb{Z}_{T_{n-1}} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} \mathbb{Z}_{T_0} \longrightarrow 0.$$

Now, for every $j \in \{0, \dots, n\}$,

$$e^j: \Delta^{n-1} \rightarrow \Delta^n, \quad \text{and} \quad T_n: \Delta^n \rightarrow \{p\}$$

so that $T_n^{(j)} = T_n \circ e^j$ is the unique function $T_{n-1}: \Delta^{n-1} \rightarrow \{p\}$. Therefore,

$$\begin{aligned} \partial_n(T_n) &= \sum_{j=0}^n (-1)^j T_n^{(j)} = \sum_{j=0}^n (-1)^j T_{n-1} = \left(\sum_{j=0}^n (-1)^j \right) T_{n-1} \\ &= \begin{cases} T_{n-1}, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Thus ∂_n is an isomorphism, if n is even, and $\partial_n = 0$, if n is odd. The singular chain complex becomes

$$\cdots \longrightarrow S_4(\{p\}) \xrightarrow{\cong} S_3(\{p\}) \xrightarrow{0} S_2(\{p\}) \xrightarrow{\cong} S_1(\{p\}) \xrightarrow{0} S_0(\{p\}) \xrightarrow{0} 0.$$

Therefore, if n is even, then $\partial_n: S_n(\{p\}) \rightarrow S_{n-1}(\{p\})$ is an isomorphism, and if n is odd, then $\partial_n = 0: S_n(\{p\}) \rightarrow S_{n-1}(\{p\})$. Thus:

(1)

$$\begin{aligned} H_0(\{p\}) &= Z_0(\{p\})/B_0(\{p\}) = S_0(\{p\})/\text{im } \partial_1 \\ &= S_0(\{p\})/0 \cong S_0(\{p\}) \cong \mathbb{Z}. \end{aligned}$$

(2) $n \geq 1$, n odd:

$$\begin{aligned} H_n(\{p\}) &= Z_n(\{p\})/B_n(\{p\}) = \ker \partial_n / \text{im} \partial_{n+1} \\ &= S_n(\{p\})/S_n(\{p\}) = 0. \end{aligned}$$

(3) $n \geq 1$, n even:

$$\begin{aligned} H_n(\{p\}) &= Z_n(\{p\})/B_n(\{p\}) = \ker \partial_n / \text{im} \partial_{n+1} \\ &= 0/0 = 0. \end{aligned}$$

Thus

$$H_n(\{p\}) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ 0, & \text{if } n > 0. \end{cases}$$

In particular we proved the following:

Theorem 14.2. (*Dimension axiom*) If X is a one-point space, then $H_n(X) = 0$, for all $n > 0$. $\square$

Proposition 14.3. Let X be path-connected, $X \neq \emptyset$. Then $H_0(X) \cong \mathbb{Z}$.

Proof. Consider

$$\cdots \longrightarrow S_1(X) \xrightarrow{\partial} S_0(X) \longrightarrow 0.$$

Here

$$Z_0(X) = S_0(X) \text{ and } H_0(X) = Z_0(X)/B_0(X) = S_0(X)/B_0(X).$$

Let $T: \Delta^0 = \{1\} \rightarrow X$ be a singular 0-simplex. We identify T with the point $T(\Delta^0) \in X$. Then an arbitrary element c_0 of $S_0(X)$ is of the form

$$c_0 = \sum n_x x,$$

where $n_x \in \mathbb{Z}$, $x \in X$, and $n_x \neq 0$, for only finitely many $x \in X$. Let then $T: \Delta^1 \rightarrow X$ be a singular 1-simplex. Then

$$\begin{aligned} \partial T &= T^{(0)} - T^{(1)} \\ &= T^{(0)}(\Delta^0) - T^{(1)}(\Delta^0) \text{ (by identification)} \\ &= T(p_1) - T(p_0), \end{aligned}$$

where $p_1 = (0, 1)$ and $p_0 = (1, 0)$.

Define a homomorphism

$$\eta: S_0(X) \rightarrow \mathbb{Z}, \quad \sum_{x \in X} n_x x \mapsto \sum_{x \in X} n_x \in \mathbb{Z}.$$

Since $X \neq \emptyset$, it follows that η is a surjection. Let $T: \Delta^1 \rightarrow X$. Then

$$\eta \partial T = \eta(T(p_1) - T(p_0)) = 1 - 1 = 0.$$

Thus $B_0(X) \subset \ker \eta$. We show that $\ker \eta \subset B_0(X)$: Let

$$c_0 = \sum_{i=1}^k n_i x_i \in \ker \eta.$$

Then $\sum_{i=1}^k n_i = 0$. Let $x_0 \in X$. For every i , let $T_i: \Delta^1 \rightarrow X$, $T_i(p_0) = x_0$ and $T_i(p_i) = x_i$. (Such T_i exist since X is path connected.) Then

$$\begin{aligned} \partial\left(\sum_{i=1}^k n_i T_i\right) &= \sum_{i=1}^k n_i \partial T_i = \sum_{i=1}^k n_i (T_i(p_i) - T_i(p_0)) \\ &= \sum_{i=1}^k n_i (x_i - x_0) = \sum_{i=1}^k n_i x_i - \underbrace{\left(\sum_{i=1}^k n_i\right) x_0}_0 \\ &= \sum_{i=1}^k n_i x_i = c_0. \end{aligned}$$

Thus $c_0 \in B_0(X)$. It follows that $\ker \eta \subset B_0(X)$, and hence that $\ker \eta = B_0(X)$. Since $\eta: S_0(X) \rightarrow \mathbb{Z}$ is a surjective homomorphism, there is an isomorphism

$$\tilde{\eta}: S_0(X)/\ker \eta \cong \mathbb{Z}.$$

Therefore, η induces an isomorphism

$$\bar{\eta}: H_0(X) = Z_0(X)/B_0(X) = S_0(X)/\ker \eta \cong \mathbb{Z}.$$

□

Definition 14.4. Let $\{G_i \mid i \in I\}$ be a family of abelian groups. The direct sum $\sum G_i$ of the groups G_i is the group of all $(g_i)_{i \in I}$ such that $g_i \neq 0$, for only finitely many $i \in I$.

Theorem 14.5. Let X be a topological space, $X \neq \emptyset$. Let $\{X_\lambda \mid \lambda \in \Lambda\}$ be the set of path components of X . Then, for every $n \geq 0$,

$$H_n(X) \cong \sum_{\lambda} H_n(X_\lambda).$$

Proof. Let $\gamma = \sum n_i T_i \in S_n(X)$. For every i , the image $T_i(\Delta^n)$ is contained in a unique path component of X . Therefore, we may write $\gamma = \sum \gamma_\lambda$, where γ_λ is the sum of the terms in γ involving the T_i for which $T_i(\Delta^n) \subset X_\lambda$.

For every $n \geq 0$, the map

$$S_n(X) \rightarrow \sum_{\lambda} S_n(X_\lambda), \quad \gamma \mapsto (\gamma_\lambda),$$

is an isomorphism. Then $\gamma \in Z_n(X)$, if and only if $\gamma_\lambda \in Z_n(X_\lambda)$, for every $\lambda \in \Lambda$, and $\gamma \in B_n(X)$, if and only if $\gamma_\lambda \in B_n(X_\lambda)$, for every $\lambda \in \Lambda$. Therefore,

$$\theta_n: H_n(X) \rightarrow \sum_{\lambda} H_n(X_\lambda), \quad \gamma B_n(X) \mapsto (\gamma_\lambda B_n(X_\lambda)),$$

is well-defined. The inverse of θ_n is

$$\phi_n: \sum_{\lambda} H_n(X_\lambda) \rightarrow H_n(X), \quad (\gamma_\lambda B_n(X_\lambda)) \mapsto \left(\sum \gamma_\lambda\right) B_n(X).$$

□

15. CHAIN COMPLEXES

Definition 15.1. A *chain complex* K is a sequence

$$\cdots \longrightarrow K_{n+1} \xrightarrow{\partial_{n+1}} K_n \xrightarrow{\partial_n} K_{n-1} \xrightarrow{\partial_{n-1}} \cdots,$$

where, for every $n \in \mathbb{Z}$, K_n is an abelian group, ∂_n is a group homomorphism, and $\partial_n \circ \partial_{n+1} = 0$.

Often $K_n = 0$, for every $n < 0$. Denote

$$Z_n(K) = \ker \partial_n \text{ and } B_n(K) = \operatorname{im} \partial_{n+1}.$$

Since $\partial_n \circ \partial_{n+1} = 0$, it follows that $B_n(K) \subset Z_n(K)$.

Definition 15.2. The n^{th} *homology group* of K is

$$H_n(K) = Z_n(K)/B_n(K).$$

Definition 15.3. Let K and L be chain complexes. A *chain map* $f: K \rightarrow L$ is a sequence of homomorphisms $\{f_n: K_n \rightarrow L_n\}$, such that the diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & K_{n+1} & \xrightarrow{\partial_{n+1}} & K_n & \xrightarrow{\partial_n} & K_{n-1} \xrightarrow{\partial_{n-1}} \cdots \\ & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} \\ \cdots & \longrightarrow & L_{n+1} & \xrightarrow{\partial'_{n+1}} & L_n & \xrightarrow{\partial'_n} & L_{n-1} \xrightarrow{\partial'_{n-1}} \cdots \end{array}$$

commutes, i.e., $\partial'_n \circ f_n = f_{n-1} \circ \partial_n$, for all $n \in \mathbb{Z}$.

Let $f: K \rightarrow L$ be a chain map. Then

$$f_n(Z_n(K)) \subset Z_n(L), \text{ and } f_n(B_n(K)) \subset B_n(L),$$

for every $n \in \mathbb{Z}$. Thus, for every $n \in \mathbb{Z}$ we obtain a homomorphism

$$H_n(f) = f_*: H_n(K) \rightarrow H_n(L), \quad [z] \mapsto [f_n(z)],$$

where $[z] = zB_n(K)$ (sometimes also denoted by $z + B_n(K)$) denotes the *homology class* of $z \in Z_n(K)$.

Definition 15.4. Chain complexes (objects) and chain maps (morphisms) form a category Comp . Composition of chain maps is defined coordinatewise: $\{g_n\} \circ \{f_n\} = \{g_n \circ f_n\}$.

Let K , L and M be chain complexes, and let $f: K \rightarrow L$ and $g: L \rightarrow M$ be chain maps. Then $g \circ f: K \rightarrow M$ is a chain map and

$$(g \circ f)_* = g_* \circ f_*: H_n(K) \rightarrow H_n(M),$$

for all $n \in \mathbb{Z}$. Clearly, the identity map $\text{id} = (\text{id}_{K_n}): K \rightarrow K$ is a chain map and it induces $H_n(\text{id}_K) = \text{id}_* = \text{id}: H_n(K) \rightarrow H_n(K)$, for every $n \in \mathbb{Z}$.

Therefore, we obtain:

Proposition 15.5. For every $n \in \mathbb{Z}$, there is a functor $H_n: \text{Comp} \rightarrow \text{Ab}$. $\square$

Definition 15.6. Let K be a chain complex. A *subcomplex* K' of K consists of subgroups $K'_n \subset K_n$ such that $\partial_n(K'_n) \subset K'_{n-1}$, for every $n \in \mathbb{Z}$.

$$\begin{array}{ccc}
 \vdots & & \vdots \\
 \downarrow & & \downarrow \\
 K'_{n+1} & \hookrightarrow & K_{n+1} \\
 \partial'_{n+1} = \partial_{n+1}|_{K'_{n+1}} \downarrow & & \downarrow \partial_{n+1} \\
 K'_n & \hookrightarrow & K_n \\
 \partial'_n = \partial_n|_{K'_n} \downarrow & & \downarrow \partial_n \\
 K'_{n-1} & \hookrightarrow & K_{n-1} \\
 \downarrow & & \downarrow \\
 \vdots & & \vdots
 \end{array}$$

Definition 15.7. Let K be a chain complex and let K' be a subcomplex of K . The *quotient complex* K/K' is the complex

$$\cdots \longrightarrow K_{n+1}/K'_{n+1} \xrightarrow{\bar{\partial}_{n+1}} K_n/K'_n \xrightarrow{\bar{\partial}_n} K_{n-1}/K'_{n-1} \xrightarrow{\bar{\partial}_{n-1}} \cdots ,$$

where $\bar{\partial}_n$ is the homomorphism induced by ∂_n :

$$\bar{\partial}_n(cK'_n) = \partial_n(c)K'_{n-1}.$$

Definition 15.8. A sequence

$$\cdots \longrightarrow G_{n+1} \xrightarrow{\partial_{n+1}} G_n \xrightarrow{\partial_n} G_{n-1} \xrightarrow{\partial_{n-1}} \cdots , \quad (*)$$

where G_n is an abelian group and ∂_n is a group homomorphism, for every $n \in \mathbb{Z}$, is called *exact*, if $\text{im} \partial_{n+1} = \ker \partial_n$ for every n .

In other words, the sequence $(*)$ is exact, if and only if it is a chain complex G such that $H_n(G) = 0$, for every n .

Definition 15.9. Let A, B, C be abelian groups, and let $\alpha: A \rightarrow B$ and $\beta: B \rightarrow C$ be group homomorphisms. The sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0 \quad (*)$$

is exact, if and only if,

- (1) α is an injection (monomorphism), and
- (2) β is a surjection (epimorphism), and
- (3) $\text{im} \alpha = \ker \beta$.

If $(*)$ is exact, it is called a *short exact sequence* of abelian groups.

Definition 15.10. Let K, L, M be chain complexes, and let $\alpha: K \rightarrow L$ and $\beta: L \rightarrow M$ be chain maps. The sequence

$$0 \longrightarrow K \xrightarrow{\alpha} L \xrightarrow{\beta} M \longrightarrow 0 \quad (**)$$

is called a *short exact sequence of chain complexes*, if

$$0 \longrightarrow K_n \xrightarrow{\alpha_n} L_n \xrightarrow{\beta_n} M_n \longrightarrow 0$$

is a short exact sequence of abelian groups, for every $n \in \mathbb{Z}$.

The chain map $\alpha: K \rightarrow L$ induces a homomorphism

$$\alpha_*: H_n(K) \rightarrow H_n(L),$$

and the chain map $\beta: L \rightarrow M$ induces a homomorphism

$$\beta_*: H_n(L) \rightarrow H_n(M),$$

for every $n \in \mathbb{Z}$. We will show that $(**)$ induces also homomorphisms

$$\Delta_n: H_n(M) \rightarrow H_{n-1}(K),$$

for every n . The homomorphisms Δ_n are called *connecting homomorphisms*.

$$\begin{array}{ccccccc} & \vdots & & \vdots & & \vdots & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & K_{n+1} & \xrightarrow{\alpha} & L_{n+1} & \xrightarrow{\beta} & M_{n+1} \longrightarrow 0 \\ & & \partial' \downarrow & & \partial \downarrow & & \partial'' \downarrow \\ 0 & \longrightarrow & K_n & \xrightarrow{\alpha} & L_n & \xrightarrow{\beta} & M_n \longrightarrow 0 \\ & & \partial' \downarrow & & \partial \downarrow & & \partial'' \downarrow \\ 0 & \longrightarrow & K_{n-1} & \xrightarrow{\alpha} & L_{n-1} & \xrightarrow{\beta} & M_{n-1} \longrightarrow 0 \\ & & \partial' \downarrow & & \partial \downarrow & & \partial'' \downarrow \\ & & \vdots & & \vdots & & \vdots \end{array}$$

First, we construct a homomorphism

$$\hat{\Delta}_n: Z_n(M) \rightarrow H_{n-1}(K)$$

as follows: Let $m \in Z_n(M)$, i.e., $m \in M_n$ and $\partial''(m) = 0$. Since β is a surjection, there is $l \in L_n$ with $\beta(l) = m$. Then

$$\beta\partial(l) = \partial''\beta(l) = \partial''(m) = 0.$$

Thus $\partial(l) \in \ker \beta$. Since the sequence

$$0 \longrightarrow K_{n-1} \xrightarrow{\alpha} L_{n-1} \xrightarrow{\beta} M_{n-1} \longrightarrow 0$$

is exact, it follows that $\partial(l) \in \text{im}\alpha$. Thus there exists $k \in K_{n-1}$ with $\alpha(k) = \partial l$. Then

$$\alpha\partial'(k) = \partial\alpha(k) = \partial\partial(l) = 0.$$

Since α is an injection, it follows that $\partial'(k) = 0$, i.e., that $k \in Z_{n-1}(K)$.

Let $l_1 \in L_n$ be another element such that $\beta(l_1) = m$. Then there is a unique $k_1 \in K_{n-1}$ with $\alpha(k_1) = \partial(l_1)$. Now

$$\beta(l - l_1) = \beta(l) - \beta(l_1) = m - m = 0.$$

Since $\ker \beta = \text{im}\alpha$, there is $\tilde{k} \in K_n$ with $\alpha(\tilde{k}) = l - l_1$. Since

$$\begin{aligned} \alpha(\partial'(\tilde{k})) &= \partial\alpha(\tilde{k}) = \partial(l - l_1) = \partial(l) - \partial(l_1) \\ &= \alpha(k) - \alpha(k_1) = \alpha(k - k_1) \end{aligned}$$

and α is an injection, it follows that $\partial'(\tilde{k}) = k - k_1$. Thus $k - k_1 \in B_{n-1}(K)$, and therefore, $[k] = [k_1] \in H_{n-1}(K)$. We obtain a well-defined function

$$\hat{\Delta}_n: Z_n(M) \rightarrow H_{n-1}(K), \quad m \mapsto [k].$$

Here $\alpha(k) = \partial l$, where $\beta(l) = m$.

Let's check that $\hat{\Delta}_n$ is a homomorphism: Let $m, m' \in Z_n(M)$. Then

$$\hat{\Delta}_n(m) = [k], \quad \text{where } \alpha(k) = \partial l, \quad \beta(l) = m,$$

and

$$\hat{\Delta}_n(m') = [k'], \quad \text{where } \alpha(k') = \partial l', \quad \beta(l') = m'.$$

Thus

$$\alpha(k + k') = \alpha(k) + \alpha(k') = \partial l + \partial l' = \partial(l + l')$$

and

$$\beta(l + l') = \beta(l) + \beta(l') = m + m'.$$

Since

$$\alpha(k + k') = \partial(l + l') \quad \text{and} \quad \beta(l + l') = m + m',$$

it follows that

$$\hat{\Delta}_n(m + m') = [k + k'] = [k] + [k'] = \hat{\Delta}_n(m) + \hat{\Delta}_n(m').$$

Hence $\hat{\Delta}_n$ is a homomorphism.

Next, let's check that $\hat{\Delta}_n(B_n(M)) = 0$: Let $m \in B_n(M) \subset M_n$. Then there is $\tilde{m} \in M_{n+1}$ such that $\partial''(\tilde{m}) = m$. Since β is a surjection, there is $\tilde{l} \in L_{n+1}$ with $\beta(\tilde{l}) = \tilde{m}$. Then

$$\beta\partial\tilde{l} = \partial''\beta\tilde{l} = \partial''\tilde{m} = m.$$

Also, $\partial\partial\tilde{l} = 0 \in L_{n-1}$. Since α is an injection, the only element in K_{n-1} that α takes to 0 is 0. Thus $\alpha(0) = \partial(\partial\tilde{l})$ and $\beta(\partial\tilde{l}) = m$. It now follows from the definition of $\hat{\Delta}_n$ that $\hat{\Delta}_n(m) = 0$. Therefore, $\hat{\Delta}_n(B_n(M)) = 0$. It follows that $\hat{\Delta}_n$ induces a homomorphism

$$\Delta_n: H_n(M) \rightarrow H_{n-1}(K), \quad [m] \mapsto [k],$$

where $\alpha(k) = l$ and $\partial l = m$.

Lemma 15.11. *Let*

$$0 \longrightarrow K \xrightarrow{\alpha} L \xrightarrow{\beta} M \longrightarrow 0 \quad (**)$$

be a short exact sequence of chain complexes. Then there is a long exact sequence

$$\cdots \longrightarrow H_{n+1}(M) \xrightarrow{\Delta} H_n(K) \xrightarrow{\alpha_*} H_n(L) \xrightarrow{\beta_*} H_n(M) \xrightarrow{\Delta} H_{n-1}(K) \longrightarrow \cdots$$

Proof. a) *Exactness at $H_n(K)$:*

(1) $\text{im } \Delta \subset \ker \alpha_*$:

Let $[m] \in H_{n+1}(M)$, $m \in Z_{n+1}(M)$. Let $l \in L_{n+1}$ be such that $\beta(l) = m$, and let $k \in Z_n(K)$ be such that $\alpha(k) = \partial l$. Then

$$\Delta([m]) = \Delta_{n+1}([m]) = [k] \in H_n(K)$$

and

$$(\alpha_* \circ \Delta)([m]) = \alpha_*[k] = [\alpha(k)] = [\partial l] = 0 \in H_n(L).$$

Therefore, $\Delta([m]) \in \ker \alpha_*$. It follows that $\text{im } \Delta \subset \ker \alpha_*$.

(2) $\ker \alpha_* \subset \text{im } \Delta$:

Assume $[k] \in H_n(K)$ is such that $\alpha_*([k]) = 0$. Then

$$[\alpha(k)] = \alpha_*([k]) = 0 \in H_n(L),$$

and hence $\alpha(k) \in B_n(L)$. Thus there exists $l \in L_{n+1}$ with $\partial l = \alpha(k)$. Let $m = \beta(l) \in M_{n+1}$. Then

$$\partial''(m) = \partial''\beta(l) = \beta\partial(l) = \beta\alpha(k) = 0,$$

since $\beta \circ \alpha = 0$. Thus $m \in Z_{n+1}(M)$. Now,

$$\beta(l) = m \in Z_{n+1}(M) \text{ and } \partial l = \alpha(k).$$

Thus

$$\hat{\Delta}(m) = [k], \text{ i.e., } \Delta([m]) = [k].$$

Hence $[k] \in \text{im } \Delta$, and it follows that $\ker \alpha_* \subset \text{im } \Delta$.

By (1) and (2), $\text{im } \Delta = \ker \alpha_*$.

b) *Exactness at $H_n(L)$:*

(1) $\text{im } \alpha_* \subset \ker \beta_*$:

$$\begin{aligned} \beta \circ \alpha = 0 &\implies 0 = 0_* = (\beta \circ \alpha)_* = \beta_* \circ \alpha_* \\ &\implies \text{im } \alpha_* \subset \ker \beta_*. \end{aligned}$$

(2) $\ker \beta_* \subset \text{im } \alpha_*$:

Let $l \in Z_n(L)$, $\beta_*([l]) = 0$. Then

$$\begin{aligned} 0 = \beta_*([l]) &= [\beta(l)] \implies \beta(l) \in B_n(M) \\ &\implies \exists \tilde{m} \in M_{n+1} : \partial''(\tilde{m}) = \beta(l). \end{aligned}$$

Since β is surjective, there exists $\tilde{l} \in L_{n+1}$ with $\beta(\tilde{l}) = \tilde{m}$. Now,

$$\begin{aligned}\beta\partial\tilde{l} &= \partial''\beta\tilde{l} = \partial''\tilde{m} = \beta(l) \\ \implies \beta(l - \partial\tilde{l}) &= \beta(l) - \beta(\partial\tilde{l}) = 0 \\ \implies l - \partial\tilde{l} &\in \ker \beta = \operatorname{im} \alpha \\ \implies \exists k \in K_n: \alpha(k) &= l - \partial\tilde{l}.\end{aligned}$$

Then

$$\begin{aligned}\alpha\partial'k &= \partial\alpha k = \partial(l - \tilde{l}) \\ &= \partial l - \underbrace{\partial\partial\tilde{l}}_0 = \partial l = 0,\end{aligned}$$

since $l \in Z_n(L)$. Since α is an injection, it follows that $\partial'k = 0$. Thus $k \in Z_n(K)$ and $[k] \in H_n(K)$. Then

$$\alpha_*([k]) = [\alpha(k)] = [l - \partial\tilde{l}] = [l].$$

Thus $[l] \in \operatorname{im} \alpha_*$, and it follows that $\ker \beta_* \subset \operatorname{im} \alpha_*$.

By (1) and (2), $\operatorname{im} \alpha_* = \ker \beta_*$.

c) *Exactness at $H_n(M)$:*

(1) $\operatorname{im} \beta_* \subset \ker \Delta$:

Assume $[l] \in H_n(L)$, where $l \in Z_n(L)$. Let $m = \beta(l)$. Then $\beta(l) = m$ and $\alpha(0) = 0 = \partial l$ imply that $\Delta([m]) = 0 \in H_{n-1}(K)$. Since $\beta_*([l]) = [\beta(l)] = [m]$, it follows that $\Delta\beta_*([l]) = \Delta([m]) = 0$. Thus $\operatorname{im} \beta_* \subset \ker \Delta$.

(2) $\ker \Delta \subset \operatorname{im} \beta_*$:

Assume $[m] \in \ker \Delta \subset H_n(M)$. Then $\Delta([m]) = [k]$, where $k \in Z_{n-1}(K)$ is such that $\alpha(k) = \partial l$ for some $l \in L_n$ and $\beta(l) = m$. Now,

$$\begin{aligned}\Delta([m]) &= [k] = 0 \Rightarrow k \in B_{n-1}(K) \\ \Rightarrow \exists \tilde{k} \in K_n: \partial'(\tilde{k}) &= k.\end{aligned}$$

Then

$$\partial\alpha(\tilde{k}) = \alpha\partial'(\tilde{k}) = \alpha(k) = \partial l.$$

Thus $\partial(l - \alpha(\tilde{k})) = 0$, i.e., $l - \alpha(\tilde{k}) \in Z_n(L)$, and

$$\beta(l - \alpha(\tilde{k})) = \beta(l) - \underbrace{\beta\alpha(\tilde{k})}_0 = \beta(l) = m.$$

Hence

$$\beta_*([l - \alpha(\tilde{k})]) = [\beta(l - \alpha(\tilde{k}))] = [m].$$

It follows that $[m] \in \operatorname{im} \beta_*$, and hence that $\ker \Delta \subset \operatorname{im} \beta_*$.

By (1) and (2), $\ker \Delta = \operatorname{im} \beta_*$. □

Proposition 15.12. *Let*

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & K & \xrightarrow{\alpha} & L & \xrightarrow{\beta} & M & \longrightarrow & 0 \\
 \downarrow & & \downarrow f & & \downarrow g & & \downarrow h & & \downarrow \\
 0 & \longrightarrow & K' & \xrightarrow{\alpha'} & L' & \xrightarrow{\beta'} & M' & \longrightarrow & 0
 \end{array}$$

be a commutative diagram, where the horizontal sequences are short exact sequences of chain complexes and f , g and h are chain maps. Then the diagram

$$\begin{array}{ccc}
 H_n(M) & \xrightarrow{\Delta} & H_{n-1}(K) \\
 h_* \downarrow & & \downarrow f_* \\
 H_n(M') & \xrightarrow{\Delta'} & H_{n-1}(K').
 \end{array}$$

commutes.

Proof. Let $[m] \in H_n(M)$. Then $m \in Z_n(M)$. Let $l \in L_n$ be such that $\beta(l) = m$, and let $k \in Z_{n-1}(K)$ be such that $\alpha(k) = \partial l$. Then

$$\Delta([m]) = [k],$$

and

$$f_*(\Delta([m])) = f_*[k] = [f(k)] \in H_{n-1}(K').$$

Now, $h_*([m]) = [h(m)] \in H_n(M')$. Also, $\beta'g(l) = h\beta(l) = h(m)$, $f(k) \in Z_{n-1}(K')$ and

$$\alpha'(f(k)) = g(\alpha(k)) = g(\partial l) = \partial g(l).$$

Thus $\beta'(g(l)) = h(m)$ and $\alpha'(f(k)) = \partial(g(l))$ imply that

$$\Delta' h_*([m]) = \Delta'([h(m)]) = [f(k)] = f_*([k]) = f_*\Delta([m]).$$

Therefore, $\Delta' \circ h_* = f_* \circ \Delta$.

□

Corollary 15.13. *The diagram*

$$\begin{array}{ccccccccc}
 \cdots & \longrightarrow & H_{n+1}(M) & \xrightarrow{\Delta} & H_n(K) & \xrightarrow{\alpha_*} & H_n(L) & \xrightarrow{\beta_*} & H_n(M) & \xrightarrow{\Delta} & H_{n-1}(K) & \longrightarrow & \cdots \\
 & & h_* \downarrow & & f_* \downarrow & & g_* \downarrow & & h_* \downarrow & & f_* \downarrow & & \\
 \cdots & \longrightarrow & H_{n+1}(M') & \xrightarrow{\Delta'} & H_n(K') & \xrightarrow{\alpha'_*} & H_n(L') & \xrightarrow{\beta'_*} & H_n(M') & \xrightarrow{\Delta'} & H_{n-1}(K') & \longrightarrow & \cdots
 \end{array}$$

commutes and the horizontal lines are exact.

Proof. The horizontal lines are exact by Lemma 15.11. The first square on the left commutes by Proposition 15.12. Since $g \circ \alpha = \alpha' \circ f$, it follows that

$$\alpha'_* \circ f_* = (\alpha' \circ f)_* = (g \circ \alpha)_* = g_* \circ \alpha_*.$$

Thus the second square from the left commutes. Since $\beta' \circ g = h \circ \beta$, it follows that

$$\beta'_* \circ g_* = (\beta' \circ g)_* = (h \circ \beta)_* = h_* \circ \beta_*.$$

Thus also the third square from the left commutes. $\square$

16. CHAIN HOMOTOPY

Definition 16.1. Let K and K' be chain complexes, and let $f: K \rightarrow K'$ and $g: K \rightarrow K'$ be chain maps. A *chain homotopy* from f to g is a family of homomorphisms

$$D_n: K_n \rightarrow K'_{n+1}$$

satisfying $\partial'_{n+1} \circ D_n + D_{n-1} \circ \partial_n = f - g$, for every n .

$$\begin{array}{ccc}
 \vdots & & \vdots \\
 \downarrow & & \downarrow \\
 K_{n+1} & \longrightarrow & K'_{n+1} \\
 \partial_{n+1} \downarrow & \nearrow D_n & \downarrow \partial'_{n+1} \\
 K_n & \xrightarrow{f, g} & K'_n \\
 \partial_n \downarrow & \nearrow D_{n-1} & \downarrow \partial'_n \\
 K_{n-1} & \longrightarrow & K'_{n-1} \\
 \downarrow & & \downarrow \\
 \vdots & & \vdots
 \end{array}$$

If there is a chain homotopy from a chain map f to a chain map g (or equivalently from g to f , since chain homotopy defines an equivalence relation on the set of chain maps), we say that f and g are *chain homotopic*. If F is a chain homotopy from f to g , we write $F: f \simeq g$.

Definition 16.2. Let K and K' be chain complexes. A chain map $f: K \rightarrow K'$ is called a *chain homotopy equivalence*, if there is a chain map $g: K' \rightarrow K$ such that $g \circ f \simeq \text{id}_K$ and $f \circ g \simeq \text{id}_{K'}$. In this case we say that K and K' are *chain homotopy equivalent*.

Proposition 16.3. If chain maps $f, g: K \rightarrow K'$ are chain homotopic, then

$$f_* = g_*: H_n(K) \rightarrow H_n(K'),$$

for every n .

Proof. Let $[k] \in H_n(K)$, where $k \in Z_n(K)$. Let $\{D_n: K_n \rightarrow K'_{n+1}\}$ be a chain homotopy from f to g . Then

$$f = g + \partial'_{n+1} \circ D_n + D_{n-1} \circ \partial_n$$

and

$$f(k) = g(k) + \partial'_{n+1}(D_n(k)) + \underbrace{D_{n-1}(\partial_n(k))}_0.$$

Thus

$$\begin{aligned} f_*([k]) &= [f(k)] = [g(k) + \partial'_{n+1}(D_n(k))] \\ &= [g(k)] + \underbrace{[\partial'_{n+1}(D_n(k))]}_0 = [g(k)] = g_*([k]). \end{aligned}$$

Therefore, $f_* = g_*$. $\square$

Let K be a chain complex. Let $\text{id}: K \rightarrow K$ and $0: K \rightarrow K$, where $\text{id}: K_n \rightarrow K_n$ is the identity map for all n , and $0: K_n \rightarrow K_n$ is the zero map for all n . Then id and 0 are chain maps.

Definition 16.4. Let K be a chain complex. If there is a chain homotopy from the identity map id of K to the zero chain map $0: K \rightarrow K$, we call this chain homotopy a *chain contraction*. If there is a chain contraction of K , then K is said to be *chain contractible*. If $H_n(K) = 0$, for all n , we say that K is *acyclic*.

Lemma 16.5. A contractible chain complex is acyclic.

Proof. Let K be a contractible chain complex. Then $\text{id}, 0: K \rightarrow K$ are chain homotopic. By Proposition 16.3, $(\text{id})_* = 0_*: H_n(K) \rightarrow H_n(K)$, for every n . However, $(\text{id})_* = \text{id}: H_n(K) \rightarrow H_n(K)$ and $0_* = 0: H_n(K) \rightarrow H_n(K)$. Thus $0 = \text{id}: H_n(K) \rightarrow H_n(K)$, for every n , which is possible only when $H_n(K) = 0$, for every n . $\square$

A chain complex K is called *free*, if K is a free abelian group for every n .

Proposition 16.6. Let K be a free chain complex. Then K is acyclic if and only if it is contractible.

Proof. By Lemma 16.5, a contractible chain complex is acyclic. Assume then that K is acyclic. Consider the following diagram:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & K_{n+1} & \xrightarrow{\partial_{n+1}} & K_n & \xrightarrow{\partial_n} & K_{n-1} \xrightarrow{\partial_{n-1}} \cdots \\ & & \text{id}_* \downarrow \downarrow 0_* & & \text{id}_* \downarrow \downarrow 0_* & & \text{id}_* \downarrow \downarrow 0_* \\ \cdots & \longrightarrow & K_{n+1} & \xrightarrow{\partial_{n+1}} & K_n & \xrightarrow{\partial_n} & K_{n-1} \xrightarrow{\partial_{n-1}} \cdots \end{array}$$

The boundary homomorphism ∂_n takes K_n onto $B_{n-1}(K)$, where $B_{n-1}(K) = Z_{n-1}(K)$, since K is acyclic. Since K_{n-1} is free, also $Z_{n-1}(K)$ is free (subgroups of free abelian groups are free abelian). Thus there is a homomorphism

$$s_{n-1}: Z_{n-1}(K) \rightarrow K_n$$

such that

$$\partial_n \circ s_{n-1} = \text{id}: Z_{n-1}(K) \rightarrow Z_{n-1}(K).$$

Then

$$\text{id}_{K_n} - s_{n-1}\partial_n: K_n \rightarrow K_n$$

maps K_n to $Z_n(K)$. Define

$$D_n: K_n \rightarrow K_{n+1}, \quad D_n = s_n \circ (\text{id}_{K_n} - s_{n-1}\partial_n).$$

Then

$$\begin{aligned} \partial_{n+1}D_n + D_{n-1}\partial_n &= \underbrace{\partial_{n+1}s_n}_{\text{id}}(\text{id}_{K_n} - s_{n-1}\partial_n) + s_{n-1}(\text{id}_{K_{n-1}} - s_{n-2}\underbrace{\partial_{n-1}}_0)\partial_n \\ &= \text{id}_{K_n} - s_{n-1}\partial_n + \underbrace{s_{n-1}\text{id}_{K_{n-1}}\partial_n}_{s_{n-1}\partial_n} \\ &= \text{id}_{K_n}. \end{aligned}$$

Thus $\{D_n\}$ is a chain contraction. $\square$

Example 16.7. Let C be the chain complex with $C_q = 0$, if $q \neq 0, 1, 2$, $C_0 = \mathbb{Z}_2$ and $C_2 = C_1 = \mathbb{Z}$. The boundary maps are defined as follows: $\partial_2(n) = 2n$, $\partial_1(n) = 0$, if n is even, and $\partial_1(n) = 1$, if n is odd. It is left as an exercise to show that C is acyclic.

Assume there is a chain contraction $D: \text{id}_K \simeq 0$, and consider the following diagram:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\partial_2} & \mathbb{Z} & \xrightarrow{\partial_1} & \mathbb{Z}_2 & \xrightarrow{\partial_0} & 0 & \longrightarrow & \cdots \\ & & \text{id} \downarrow & & D_2 \swarrow & & \text{id} \downarrow & & D_1 \swarrow & & \text{id} \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\partial_2} & \mathbb{Z} & \xrightarrow{\partial_1} & \mathbb{Z}_2 & \xrightarrow{\partial_0} & 0 & \longrightarrow & \cdots \\ & & & & & & \text{id} \downarrow & & D_0 \swarrow & & \text{id} \downarrow & & \\ & & & & & & & & \text{id} \downarrow & & D_{-1} \swarrow & & \\ & & & & & & & & & & \text{id} \downarrow & & \end{array}$$

Then

$$\underbrace{D_{-1}\partial_0}_0 + \partial_1 D_0 = \text{id}: \mathbb{Z}_2 \rightarrow \mathbb{Z}_2.$$

Thus

$$\mathbb{Z}_2 \xrightarrow{D_0} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z}_2$$

equals the identity homomorphism $\mathbb{Z}_2 \rightarrow \mathbb{Z}_2$. This is impossible since the only homomorphism $\mathbb{Z}_2 \rightarrow \mathbb{Z}$ is trivial. It follows that C is not contractible.

17. RELATIVE HOMOLOGY GROUPS

Let X be a topological space and let $S_*(X)$ be the singular chain complex of X . Let $A \subset X$ and let $i: A \hookrightarrow X$ be the inclusion. Then i induces a chain map $i_\#: S_*(A) \rightarrow S_*(X)$. For every n , the map

$$i_\#: S_n(A) \rightarrow S_n(X), \quad \sum_T n_T T \mapsto \sum_T n_T (i \circ T),$$

is injective. Now, $S_*(A)$ is a subcomplex of $S_*(X)$, and

$$S_n(A) = \left\{ \sum_{r=1}^k n_r T_r \in S_n(X) \mid T_r(\Delta^n) \subset A \forall r \right\}.$$

The corresponding quotient complex is $S_*(X)/S_*(A)$, where

$$(S_*(X)/S_*(A))_n = S_n(X)/S_n(A).$$

The complex $S_*(X)/S_*(A)$ is called the *singular chain complex of the pair* (X, A) . We obtain a short exact sequence of chain complexes:

$$0 \rightarrow S_*(A) \xrightarrow{i_\#} S_*(X) \xrightarrow{j_\#} S_*(X)/S_*(A) \rightarrow 0.$$

Thus, for every n , the sequence

$$0 \rightarrow S_n(A) \xrightarrow{i_\#} S_n(X) \xrightarrow{j_\#} S_n(X)/S_n(A) \rightarrow 0$$

is exact.

Notice that,

$$S_n(X) = \sum_{T \in \text{Map}(\Delta^n, X)} \mathbb{Z}_T = \sum_{T \in \text{Map}_1} \mathbb{Z}_T \oplus \sum_{T \in \text{Map}_2} \mathbb{Z}_T,$$

where

$$\text{Map}_1 = \{T \in \text{Map}(\Delta^n, X) \mid T(\Delta^n) \subset A\}$$

and

$$\text{Map}_2 = \{T \in \text{Map}(\Delta^n, X) \mid T(\Delta^n) \cap (X \setminus A) \neq \emptyset\}.$$

Thus

$$S_n(X)/S_n(A) \cong \sum_{T \in \text{Map}_2} \mathbb{Z}_T,$$

i.e., $S_n(X)/S_n(A)$ is a free abelian group for every n .

Definition 17.1. The n^{th} *relative homology group* $H_n(X, A)$ of the pair (X, A) is the n^{th} homology group $H_n(S_*(X)/S_*(A))$ of the quotient complex $S_*(X)/S_*(A)$.

According to Lemma 15.11, the short exact sequence

$$0 \rightarrow S_*(A) \xrightarrow{i_\#} S_*(X) \xrightarrow{j_\#} S_*(X)/S_*(A) \rightarrow 0$$

induces a long exact sequence in homology:

$$\cdots \longrightarrow H_{n+1}(X, A) \xrightarrow{\Delta} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\Delta} H_{n-1}(A) \longrightarrow \cdots$$

This sequence is called the *exact homology sequence* of the pair (X, A) , see the Eilenberg-Steenrod Axioms.

Let (X, A) and (Y, B) be topological pairs and let $f: (X, A) \rightarrow (Y, B)$ be a continuous map. Thus $f: X \rightarrow Y$ is continuous and $f(A) \subset B$. Let $f|: A \rightarrow B$ denote the restriction. Then f induces the map

$$f_\#: S_*(X) \rightarrow S_*(Y),$$

and $f_\#(S_n(A)) \subset S_n(B)$, for every n . Therefore, we obtain

$$f_\#: S_n(X)/S_n(A) \rightarrow S_n(Y)/S_n(B),$$

and the diagram

$$\begin{array}{ccccc}
 S_n(A) & \xrightarrow{i_\#} & S_n(X) & \xrightarrow{j_\#} & S_n(X)/S_n(A) \\
 f_\# \downarrow & & f_\# \downarrow & & f_\# \downarrow \\
 S_n(B) & \xrightarrow{i'_\#} & S_n(Y) & \xrightarrow{j'_\#} & S_n(Y)/S_n(B)
 \end{array}$$

commutes, for every n . The chain map

$$f_\# : S_n(X)/S_n(A) \rightarrow S_n(Y)/S_n(B)$$

induces

$$f_* : H_n(X, A) \rightarrow H_n(Y, B),$$

for every n . Let $g : (Y, B) \rightarrow (Z, C)$ be a continuous function. Then

$$(g \circ f)_* = g_* \circ f_* : H_n(X, A) \rightarrow H_n(Z, C).$$

The identity map $\text{id} : (X, A) \rightarrow (X, A)$ induces

$$\text{id}_* = \text{id} : H_n(X, A) \rightarrow H_n(X, A),$$

for every n .

Remark 17.2. Consider the case where $A = \emptyset$: We define $S_n(\emptyset) = \{0\}$, for every n . Then, for every n ,

$$S_n(X)/S_n(\emptyset) = S_n(X)/\{0\} = S_n(X),$$

and

$$H_n(X, \emptyset) = H_n(S_n(X)/S_n(\emptyset)) = H_n(X).$$

A continuous function $f : (X, A) \rightarrow (Y, B)$ induces the commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & S_*(A) & \xrightarrow{i_\#} & S_*(X) & \xrightarrow{j_\#} & S_*(X)/S_*(A) \longrightarrow 0 \\
 & & f_\# \downarrow & & f_\# \downarrow & & f_\# \downarrow \\
 0 & \longrightarrow & S_*(B) & \xrightarrow{i'_\#} & S_*(Y) & \xrightarrow{j'_\#} & S_*(Y)/S_*(B) \longrightarrow 0,
 \end{array}$$

where the horizontal lines are exact sequences. By Corollary 15.13, the diagram ($\heartsuit$)

$$\begin{array}{ccccccccc}
 \cdots & \longrightarrow & H_{n+1}(X, A) & \xrightarrow{\Delta} & H_n(A) & \xrightarrow{i_*} & H_n(X) & \xrightarrow{j_*} & H_n(X, A) & \xrightarrow{\Delta} & H_{n-1}(A) & \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 \cdots & \longrightarrow & H_{n+1}(Y, B) & \xrightarrow{\Delta'} & H_n(B) & \xrightarrow{i'_*} & H_n(Y) & \xrightarrow{j'_*} & H_n(Y, B) & \xrightarrow{\Delta'} & H_{n-1}(B) & \longrightarrow \cdots
 \end{array}$$

commutes and the horizontal lines are long exact sequences. Since the squares having Δ and Δ' are commutative, it follows that singular homology theory satisfies Eilenberg - Steenrod axiom **A3**.

Proposition 17.3. *Let $f: (X, A) \rightarrow (Y, B)$ be a continuous function. Assume $(f|A)_*: H_n(A) \rightarrow H_n(B)$ and $f_*: H_n(X) \rightarrow H_n(Y)$ are isomorphisms, for every n . Then also*

$$f_*: H_n(X, A) \rightarrow H_n(Y, B)$$

is an isomorphism, for every n .

Proof. The proof follows immediately from the fact that the diagram ($\heartsuit$) commutes and from the following 5-lemma. $\square$

Lemma 17.4. (*5-lemma*) *Let*

$$\begin{array}{ccccccccc} C_1 & \xrightarrow{\alpha_1} & C_2 & \xrightarrow{\alpha_2} & C_3 & \xrightarrow{\alpha_3} & C_4 & \xrightarrow{\alpha_4} & C_5 \\ f_1 \downarrow & & f_2 \downarrow & & f_3 \downarrow & & f_4 \downarrow & & f_5 \downarrow \\ D_1 & \xrightarrow{\beta_1} & D_2 & \xrightarrow{\beta_2} & D_3 & \xrightarrow{\beta_3} & D_4 & \xrightarrow{\beta_4} & D_5 \end{array}$$

be a commutative diagram of (not necessarily abelian) groups and homomorphisms. Assume the horizontal lines are exact. (This means that the top line is exact at C_2, C_3 and C_4 , and the bottom line is exact at D_2, D_3 and D_4 .) Then the following hold:

- (1) *If f_2 and f_4 are surjective and f_5 is injective, then f_3 is surjective.*
- (2) *If f_2 and f_4 are injective and f_1 is surjective, then f_3 is injective.*
- (3) *If f_1, f_2, f_4 and f_5 are isomorphisms, then also f_3 is an isomorphism.*

Proof. Assume first that f_2 and f_4 are surjective and f_5 is injective. We show that f_3 is surjective. Let $d_3 \in D_3$. Then $\beta_3(d_3) \in D_4$. Since f_4 is surjective, there is $c_4 \in C_4$ with $f_4(c_4) = \beta_3(d_3)$. Then

$$f_5(\alpha_4(c_4)) = \beta_4(f_4(c_4)) = \beta_4(\beta_3(d_3)) = 0,$$

since $\beta_4 \circ \beta_3 = 0$. Since f_5 is an injection, it follows that $\alpha_4(c_4) = 0$. Thus $c_4 \in \ker \alpha_4 = \operatorname{im} \alpha_3$, and it follows that there is $c_3 \in C_3$ such that $\alpha_3(c_3) = c_4$. Then

$$\begin{aligned} \beta_3(f_3(c_3) - d_3) &= \beta_3(f_3(c_3)) - \beta_3(d_3) \\ &= f_4(\alpha_3(c_3)) - \beta_3(d_3) \\ &= f_4(c_4) - \beta_3(d_3) = 0. \end{aligned}$$

Thus $f_3(c_3) - d_3 \in \ker \beta_3 = \operatorname{im} \beta_2$. Hence there is $d_2 \in D_2$ such that $\beta_2(d_2) = f_3(c_3) - d_3$. Since f_2 is surjective, there is $c_2 \in C_2$ such that $f_2(c_2) = d_2$. Then

$$f_3(\alpha_2(c_2)) = \beta_2(f_2(c_2)) = \beta_2(d_2) = f_3(c_3) - d_3.$$

Therefore,

$$d_3 = f_3(c_3) - f_3(\alpha_2(c_2)) = f_3(c_3 - \alpha_2(c_2)).$$

Thus $d_3 \in \operatorname{im} f_3$, and it follows that f_3 is surjective.

Assume then that f_2 and f_4 are injective and f_1 is surjective. We show that f_3 is injective. Let $c_3 \in C_3$. Assume $f_3(c_3) = 0$. Then $f_4\alpha_3(c_3) - \beta_3f_3(c_3) = 0$. Since

f_4 is an injection, it follows that $\alpha_3(c_3) = 0$. Thus $c_3 \in \ker \alpha_3 = \operatorname{im} \alpha_2$. But then there is $c_2 \in C_2$ such that $\alpha_2(c_2) = c_3$. It follows that

$$\beta_2 f_2(c_2) = f_3 \alpha_2(c_2) = f_3(c_3) = 0.$$

Thus $f_2(c_2) \in \ker \beta_2 = \operatorname{im} \beta_1$, and there exists $d_1 \in D_1$ such that $\beta_1(d_1) = f_2(c_2)$. Since f_1 is surjective, there exists $c_1 \in C_1$ such that $f_1(c_1) = d_1$. Then

$$f_2 \alpha_1(c_1) = \beta_1 f_1(c_1) = \beta_1(d_1) = f_2(c_2).$$

Since f_2 is an injection, it follows that $\alpha_1(c_1) = c_2$. Since $\alpha_1(c_1) = c_2$ and $\alpha_2(c_2) = c_3$, it follows that

$$0 = \underbrace{\alpha_2 \alpha_1}_{0}(c_1) = \alpha_2(c_2) = c_3.$$

Therefore, f_3 is an injection.

Part (3) follows immediately from parts (1) and (2). $\square$

Example 17.5. Let X be a non-empty convex subset of $\mathbb{R}^m$, $m \geq 0$. Then

$$\begin{cases} H_0(X) \cong \mathbb{Z}, \\ H_p(X) = 0, \text{ if } p > 0. \end{cases}$$

Proof. Recall that a subset $X \subset \mathbb{R}^m$ is *convex*, if and only if

$$\{(1-t)x + ty \mid x, y \in X, 0 \leq t \leq 1\} \subset X.$$

If X is convex, then it is path-connected, and by Proposition 14.3, $H_0(X) \cong \mathbb{Z}$. Let $x_0 \in X$. For a continuous function $T: \Delta^n \rightarrow X$, define

$$x_0 \cdot T: \Delta^{n+1} \rightarrow X$$

by setting

$$(x_0 \cdot T)(t_0, \dots, t_{n+1}) = \begin{cases} x_0, & \text{if } t_0 = 1, \\ t_0 x_0 + (1-t_0)T\left(\frac{t_1}{1-t_0}, \dots, \frac{t_{n+1}}{1-t_0}\right), & \text{if } 0 \leq t_0 < 1. \end{cases}$$

Notice that

$$x_0 \in X \text{ and } T\left(\frac{t_1}{1-t_0}, \dots, \frac{t_{n+1}}{1-t_0}\right) \in X.$$

Since X is convex, it follows that

$$t_0 x_0 + (1-t_0)T\left(\frac{t_1}{1-t_0}, \dots, \frac{t_{n+1}}{1-t_0}\right) \in X,$$

and thus that $x_0 \cdot T$ is well-defined.

We next check that $x_0 \cdot T$ is continuous: Clearly, $x_0 \cdot T$ is continuous at points $(t_0, \dots, t_{n+1})$, where $t_0 \neq 1$. Let's check that $x_0 \cdot T$ is continuous at $(1, 0, \dots, 0)$. Notice that $(x_0 \cdot T)(1, 0, \dots, 0) = x_0$. For $t_0 < 1$,

$$\begin{aligned} \|(x_0 \cdot T)(t_0, \dots, t_{n+1}) - x_0\| &= \|(1-t_0)T\left(\frac{t_1}{1-t_0}, \dots, \frac{t_{n+1}}{1-t_0}\right) + (t_0-1)x_0\| \\ &\leq (1-t_0)\left(\|T\left(\frac{t_1}{1-t_0}, \dots, \frac{t_{n+1}}{1-t_0}\right)\| + \|x_0\|\right). \end{aligned}$$

Since Δ^n is compact and T is continuous, it follows that $T(\Delta^n)$ is compact. In particular, $T(\Delta^n)$ is bounded. Therefore, there exists $M > 0$ satisfying

$$\|T(\frac{t_1}{1-t_0}, \dots, \frac{t_{n+1}}{1-t_0})\| + \|x_0\| \leq M,$$

for every $(\frac{t_1}{1-t_0}, \dots, \frac{t_{n+1}}{1-t_0}) \in \Delta^n$. Thus

$$(1-t_0) \underbrace{\left(\|T(\frac{t_1}{1-t_0}, \dots, \frac{t_{n+1}}{1-t_0})\| + \|x_0\| \right)}_{< M} \rightarrow 0 \text{ when } t_0 \rightarrow 1.$$

Thus $x_0 \cdot T$ is continuous also at $(1, 0, \dots, 0)$ and it follows that $x_0 \cdot T$ is continuous.

Define

$$(x_0, \cdot): S_n(X) \rightarrow S_{n+1}(X), \quad T \mapsto x_0 \cdot T,$$

and extend linearly,

$$(x_0, \cdot) \left(\sum_{i=1}^k n_i T_i \right) = \sum_{i=1}^k n_i (x_0 \cdot T_i).$$

Then $(x_0, \cdot)$ is a homomorphism.

We then determine the boundary $\partial(x_0 \cdot T)$: Since $x_0 \cdot T \in S_{n+1}(X)$ we need to figure out the

$$(x_0 \cdot T) \circ e^j: \Delta^n \rightarrow X,$$

where

$$((x_0 \cdot T) \circ e^j)(t_0, t_1, \dots, t_n) = (x_0 \cdot T)(t_0, \dots, t_{j-1}, 0, t_j, \dots, t_n),$$

for $0 \leq j \leq n+1$.

Assume first that $n = 0$. For $j = 0$,

$$((x_0 \cdot T) \circ e^0)(1 = t_0) = (x_0 \cdot T)(0, t_0) = (x_0 \cdot T)(0, 1) = 0 \cdot x_0 + 1 \cdot T(1) = T(1).$$

Thus $(x_0 \cdot T) \circ e^0 = T$. For $j = 1$,

$$((x_0 \cdot T) \circ e^1)(1) = (x_0 \cdot T)(1, 0) = x_0.$$

Assume then that $n \geq 1$. For $j = 0$,

$$((x_0 \cdot T) \circ e^0)(t_0, \dots, t_n) = (x_0 \cdot T)(0, t_0, \dots, t_n) = T(t_0, \dots, t_n).$$

Thus $(x_0 \cdot T) \circ e^0 = T$. For $1 \leq j \leq n+1$,

$$\begin{aligned} & ((x_0 \cdot T) \circ e^j)(t_0, \dots, t_n) = (x_0 \cdot T)(t_0, \dots, t_{j-1}, 0, t_j, \dots, t_n) \\ &= \begin{cases} x_0, & \text{if } t_0 = 1, \\ t_0(x_0) + (1-t_0)T(\underbrace{\frac{t_1}{1-t_0}, \dots, \frac{t_n}{1-t_0}}_{\text{the } j^{\text{th}} \text{ coord. } = 0}), & \text{if } 0 \leq t_0 < 1. \end{cases} \end{aligned}$$

Also,

$$\begin{aligned}
& (x_0 \cdot (Te^{j-1}))(t_0, \dots, t_n) \\
&= \begin{cases} x_0, & \text{if } t_0 = 1, \\ t_0(x_0) + (1 - t_0)Te^{j-1}\left(\frac{t_1}{1-t_0}, \dots, \frac{t_n}{1-t_0}\right), & \text{if } 0 \leq t_0 < 1, \end{cases} \\
&= \begin{cases} x_0, & \text{if } t_0 = 1, \\ t_0(x_0) + (1 - t_0)T\left(\frac{t_1}{1-t_0}, \dots, \frac{t_{j-1}}{1-t_0}, 0, \frac{t_j}{1-t_0}, \dots, \frac{t_n}{1-t_0}\right), & \text{if } 0 \leq t_0 < 1. \end{cases}
\end{aligned}$$

Thus $(x_0 \cdot T) \circ e^j = x_0 \cdot (Te^{j-1})$. Therefore, for $n \geq 1$,

$$\begin{aligned}
\partial \circ (x_0, \cdot)(T) &= \partial(x_0 \cdot T) = \sum_{j=0}^{n+1} (-1)^j (x_0 \cdot T) \circ e^j \\
&= (x_0 \cdot T) \circ e^0 + \sum_{j=1}^{n+1} (-1)^j \underbrace{(x_0 \cdot T) \circ e^j}_{x_0 \cdot (Te^{j-1})} \\
&= \underbrace{(x_0 \cdot T) \circ e^0}_T - \sum_{i=0}^n (-1)^i \underbrace{x_0 \cdot (T \circ e^i)}_{(x_0, \cdot)(T \circ e^i)} \quad (i = j - 1) \\
&= T - (x_0, \cdot) \left(\underbrace{\sum_{i=0}^n (-1)^i T \circ e^i}_{\partial T} \right) \\
&= T - (x_0, \cdot)(\partial T).
\end{aligned}$$

Therefore, for $n \geq 1$,

$$\partial \circ (x_0, \cdot) = \text{id} - (x_0, \cdot) \circ \partial.$$

We show that $Z_n(X) = B_n(X)$, for every $n \geq 1$. Let $c \in Z_n(X)$. Then $\partial c = 0$. Thus

$$\partial \circ (x_0, \cdot)(c) + \underbrace{(x_0, \cdot)\partial c}_0 = \text{id}(c) = c,$$

and it follows that

$$c = \partial \left(\underbrace{(x_0, \cdot)(c)}_{\in S_{n+1}(X)} \right).$$

Thus $c \in B_n(X)$. Consequently,

$$H_n(X) = Z_n(X)/B_n(X) = 0,$$

for every $n \geq 1$. □

18. HOMOTOPY INVARIANCE OF HOMOLOGY

Recall that two functions $f, g: (X, A) \rightarrow (Y, B)$ are called homotopic if there is a homotopy $F: X \times I \rightarrow Y$ such that $F_0 = f$, $F_1 = g$ and $F(a, t) \in B$, for every $(a, t) \in A \times I$. In this section we prove the following result:

Theorem 18.1. *Let $f, g: (X, A) \rightarrow (Y, B)$ be homotopic functions. Then the induced homomorphisms $f_{\#}, g_{\#}: S_*(X, A) \rightarrow S_*(Y, B)$ are chain homotopic.*

Proposition 16.3 and Theorem 18.1 imply

Theorem 18.2. *If $f, g: (X, A) \rightarrow (Y, B)$ are homotopic, then*

$$f_* = g_*: H_n(X, A) \rightarrow H_n(Y, B),$$

for every n . □

Theorem 18.3. *If $f: X \rightarrow Y$ is a homotopy equivalence, then*

$$f_*: H_n(X) \rightarrow H_n(Y)$$

is an isomorphism, for every n .

Proof. Let $g: Y \rightarrow X$ be a continuous map such that $g \circ f \simeq \text{id}_X$ and $f \circ g \simeq \text{id}_Y$. By Theorem 18.2,

$$g_* \circ f_* = (g \circ f)_* = (\text{id}_X)_* = \text{id}: H_n(X) \rightarrow H_n(X), \text{ for every } n,$$

and

$$f_* \circ g_* = (f \circ g)_* = (\text{id}_Y)_* = \text{id}: H_n(Y) \rightarrow H_n(Y), \text{ for every } n.$$

Therefore, f_* is an isomorphism with the inverse $(f_*)^{-1} = g_*$, for every n . □

Remark 18.4. Theorem 18.2 says that Eilenberg - Steenrod axiom 5 (the Homotopy axiom) holds for singular homology.

We now return to the proof of Theorem 18.1. There exist different proofs for the result, we will do a proof based on induction.

Proof of Theorem 18.1. We will first show that if $f \simeq g: X \rightarrow Y$, then $f_{\#}, g_{\#}: S_*(X) \rightarrow S_*(Y)$ are chain homotopic. The chain homotopy that we will construct has a certain naturality property that then implies the claim in the relative case. Define

$$i_0: X \rightarrow X \times I, \quad x \mapsto (x, 0),$$

and

$$i_1: X \rightarrow X \times I, \quad x \mapsto (x, 1).$$

Then

$$\text{id}: X \times I \rightarrow X \times I$$

is a homotopy $i_0 \simeq i_1$.

Claim 1: To show that $f_{\#}, g_{\#}: S_*(X) \rightarrow S_*(Y)$ are chain homotopic, it suffices to prove that $(i_0)_{\#}, (i_1)_{\#}: S_*(X) \rightarrow S_*(X \times I)$ are chain homotopic.

Proof of Claim 1: Let $F: X \times I \rightarrow Y, F: f \simeq g$. Assume $D: S_*(X) \rightarrow S_*(X \times I)$ is a chain homotopy from $(i_0)_{\#}$ to $(i_1)_{\#}$:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & S_{n+2}(X) & \xrightarrow{\partial_{n+2}} & S_{n+1}(X) & \xrightarrow{\partial_{n+1}} & S_n(X) & \xrightarrow{\partial_n} & S_{n-1}(X) & \longrightarrow & \cdots \\
& & \downarrow & \nearrow D_{n+1} & \downarrow & \nearrow D_n & \downarrow & \nearrow D_{n-1} & \downarrow (i_0)_\#, (i_1)_\# & & \\
\cdots & \longrightarrow & S_{n+2}(X \times I) & \xrightarrow{\partial'_{n+2}} & S_{n+1}(X \times I) & \xrightarrow{\partial'_{n+1}} & S_n(X \times I) & \xrightarrow{\partial'_n} & S_{n-1}(X \times I) & \longrightarrow & \cdots
\end{array}$$

where

$$\partial'_{n+1}D_n + D_{n-1}\partial_n = (i_1)_\# - (i_0)_\#.$$

Let

$$F_\# : S_*(X \times I) \rightarrow S_*(Y)$$

be the chain map induced by $F : X \times I \rightarrow Y$. The following diagram commutes:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & S_{n+2}(X \times I) & \xrightarrow{\partial'_{n+2}} & S_{n+1}(X \times I) & \xrightarrow{\partial'_{n+1}} & S_n(X \times I) & \xrightarrow{\partial'_n} & S_{n-1}(X \times I) & \longrightarrow & \cdots \\
& & \downarrow F_\# & & \downarrow F_\# & & \downarrow F_\# & & \downarrow F_\# & & \\
\cdots & \longrightarrow & S_{n+2}(Y) & \xrightarrow{\partial''_{n+2}} & S_{n+1}(Y) & \xrightarrow{\partial''_{n+1}} & S_n(Y) & \xrightarrow{\partial''_n} & S_{n-1}(Y) & \longrightarrow & \cdots
\end{array}$$

Then

$$\begin{aligned}
F_\#(\partial'D + D\partial) &= F_\#((i_1)_\# - (i_0)_\#) \\
&= F_\#(i_1)_\# - F_\#(i_0)_\# \\
&= (F \circ i_1)_\# - (F \circ i_0)_\# \\
&= g_\# - f_\#.
\end{aligned}$$

Since $F_\#$ is a chain map, it follows that

$$\partial''F_\# = F_\#\partial'.$$

Therefore,

$$\begin{aligned}
\partial''(F_\#D) + (F_\#D)\partial &= F_\#\partial'D + F_\#D\partial \\
&= F_\#(\partial'D + D\partial) \\
&= g_\# - f_\#.
\end{aligned}$$

Thus $F_\# \circ D : S_*(X) \rightarrow S_*(Y)$ is a chain homotopy from $g_\#$ to $f_\#$. This proves Claim 1.

We next prove that $(i_0)_\#$ and $(i_1)_\#$ are chain homotopic. In fact, we prove a stronger result:

Claim 2: For every topological space X , there is a chain homotopy $D_X : S_*(X) \rightarrow S_*(X \times I)$ from $(i_1)_\#$ to $(i_0)_\#$ satisfying the following *naturality condition*: If

$h: X \rightarrow X'$ is a continuous function, then the diagram

$$\begin{array}{ccc} S_n(X) & \xrightarrow{D_X} & S_{n+1}(X \times I) \\ h_{\#} \downarrow & & \downarrow (h \times \text{id})_{\#} \\ S_n(X') & \xrightarrow{D_{X'}} & S_{n+1}(X' \times I) \end{array}$$

commutes, for every n .

Proof of Claim 2: The proof is done by induction on n .

First, let $n = 0$. Let $T: \Delta^0 \rightarrow X$ be a singular 0-simplex of X . Define a singular 1-simplex $D_X T$ of $X \times I$ by

$$D_X T: \Delta^1 \rightarrow X \times I, \quad (t_0, t_1) \mapsto \underbrace{(T(\Delta^0), t_1)}_{T(1)}.$$

(Here $t_0 + t_1 = 1$, $t_0, t_1 \geq 0$, and $\Delta^0 = \{1\}$.) We will determine the boundary of $D_X T$: Here

$$(D_X T) \circ e^0: \Delta^0 = \{1\} \rightarrow X \times I,$$

$$(D_X T)e^0(1) = (D_X T)(0, 1) = (T(\Delta^0), 1) = (i_1 \circ T)(1)$$

and

$$(D_X T)e^1(1) = (D_X T)(1, 0) = (T(\Delta^0), 0) = (i_0 \circ T)(1).$$

Thus

$$\begin{aligned} \partial'(D_X T) &= (D_X T) \circ e^0 - (D_X T) \circ e^1 = i_1 \circ T - i_0 \circ T \\ &= (i_1)_{\#}(T) - (i_0)_{\#}(T) = ((i_1)_{\#} - (i_0)_{\#})(T). \end{aligned}$$

The homomorphisms can be seen in the following diagram,

$$\begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ S_1(X) & \xrightarrow{\quad} & S_1(X \times I) \\ \partial \downarrow & \nearrow D_X & \downarrow \partial' \\ S_0(X) & \xrightarrow[(i_1)_{\#}]{(i_0)_{\#}} & S_0(X \times I) \\ \downarrow & \nearrow 0 & \downarrow \\ \underbrace{S_{-1}(X)}_0 & & \underbrace{S_{-1}(X \times I)}_0 \end{array}$$

where

$$\partial' D_X + \underbrace{D_X \partial}_0 = \partial' D_X = (i_1)_{\#} - (i_0)_{\#}.$$

The naturality condition is also satisfied: Let $h: X \rightarrow X'$ be a continuous function. On one hand,

$$((h \times \text{id})_{\#} D_X)(T) = (h \times \text{id})_{\#}(D_X T) = (h \times \text{id}) \circ D_X T: \Delta^1 \rightarrow X' \times I.$$

Thus

$$\begin{aligned} ((h \times \text{id})_{\#} D_X T)(t_0, t_1) &= (h \times \text{id})(D_X T(t_0, t_1)) \\ &= (h \times \text{id})(T(\Delta^0), t_1) \\ &= (hT(\Delta^0), t_1). \end{aligned}$$

On the other hand,

$$(D_{X'} h_{\#})(T) = D_{X'}(h \circ T).$$

Therefore,

$$(D_{X'} h_{\#})(T)(t_0, t_1) = D_{X'}(h \circ T)(t_0, t_1) = (hT(\Delta^0), t_1).$$

Thus

$$D_{X'} \circ h_{\#} = (h \times \text{id})_{\#} \circ D_X,$$

and it follows that the naturality condition holds for $n = 0$. Consequently, Claim 2 holds for $n = 0$.

Induction assumption: Let $n \geq 1$. Assume that $(D_X)_j: S_j(X) \rightarrow S_{j+1}(X \times I)$ has been constructed for every $j < n$ in such a way that both the chain homotopy condition and the naturality condition holds.

We next construct $(D_X)_n: S_n(X) \rightarrow S_{n+1}(X \times I)$. Since the naturality condition must hold, the diagram

$$\begin{array}{ccc} S_n(X) & \xrightarrow{D_X} & S_{n+1}(X \times I) \\ h_{\#} \downarrow & & \downarrow (h \times \text{id})_{\#} \\ S_n(X') & \xrightarrow{D_{X'}} & S_{n+1}(X' \times I) \end{array}$$

must commute for every X' and for every continuous $h: X \rightarrow X'$. Choose $X = \Delta^n$, $X' = X$ and let $h = T: \Delta^n \rightarrow X$. Then the diagram ($\spadesuit$)

$$\begin{array}{ccc} S_n(\Delta^n) & \xrightarrow{D_{\Delta^n}} & S_{n+1}(\Delta^n \times I) \\ T_{\#} \downarrow & & \downarrow (T \times \text{id})_{\#} \\ S_n(X) & \xrightarrow{D_X} & S_{n+1}(X \times I) \end{array}$$

must commute.

Let $\text{id}_{\Delta^n}: \Delta^n \rightarrow \Delta^n$ be the identity map. Then $\text{id}_{\Delta^n} \in S_n(\Delta^n)$. Now,

$$T_{\#}(\text{id}_{\Delta^n}) = T \circ \text{id}_{\Delta^n} = T \in S_n(X).$$

Since the diagram ($\spadesuit$) must commute, it follows that

$$(T \times \text{id})_{\#} D_{\Delta^n}(\text{id}_{\Delta^n}) = D_X T_{\#}(\text{id}_{\Delta^n}),$$

i.e.,

$$D_X T = D_X T_{\#}(\text{id}_{\Delta^n}) = (T \times \text{id})_{\#} D_{\Delta^n}(\text{id}_{\Delta^n}).$$

This means that D_X is completely determined when we first construct $D_{\Delta^n}(\text{id}_{\Delta^n})$:

$$D_{\Delta^n}: S_n(\Delta^n) \rightarrow S_{n+1}(\Delta^n \times I), \quad \text{id}_{\Delta^n} \mapsto ?.$$

Let's consider certain element $c \in S_n(\Delta^n \times I)$:

$$c = (i_1)_{\#}(\text{id}_{\Delta^n}) - (i_0)_{\#}(\text{id}_{\Delta^n}) - D_{\Delta^n}(\partial(\text{id}_{\Delta^n})).$$

Notice that $\text{id}_{\Delta^n} \in S_n(\Delta^n)$, so $\partial(\text{id}_{\Delta^n}) \in S_{n-1}(\Delta^n)$, and since by the induction assumption $D_{\Delta^n}: S_{n-1}(\Delta^n) \rightarrow S_n(\Delta^n \times I)$ has been constructed, it follows that c indeed is an element in $S_n(\Delta^n \times I)$.

We next check that c is a cycle:

$$\begin{aligned} \partial c &= \partial(i_1)_{\#}(\text{id}_{\Delta^n}) - \partial(i_0)_{\#}(\text{id}_{\Delta^n}) - \partial D_{\Delta^n}(\partial(\text{id}_{\Delta^n})) \\ &= (i_1)_{\#} \partial(\text{id}_{\Delta^n}) - (i_0)_{\#} \partial(\text{id}_{\Delta^n}) \\ &\quad - \underbrace{\left((i_1)_{\#} \partial(\text{id}_{\Delta^n}) - (i_0)_{\#} \partial(\text{id}_{\Delta^n}) - \overbrace{D_{\Delta^n} \partial(\partial(\text{id}_{\Delta^n}))}^{0, \text{ since } \partial \partial = 0} \right)}_{= \partial D_{\Delta^n}(\partial(\text{id}_{\Delta^n}))} \\ &= 0. \end{aligned}$$

Thus $c \in S_n(\Delta^n \times I)$ is such that $\partial c = 0 \in S_{n-1}(\Delta^n \times I)$. The space $\Delta^n \times I \subset \mathbb{R}^{n+1} \times I \subset \mathbb{R}^{n+2}$ is convex. By Example 17.5, $H_m(\Delta^n \times I) = 0$, for every $m \geq 1$. In particular, $H_n(\Delta^n \times I) = 0$. Thus $c \in Z_n(\Delta^n \times I) = B_n(\Delta^n \times I)$ and it follows that there is $b \in S_{n+1}(\Delta^n \times I)$ such that $\partial b = c$. But then

$$\partial b + D_{\Delta^n}(\partial(\text{id}_{\Delta^n})) = (i_1)_{\#}(\text{id}_{\Delta^n}) - (i_0)_{\#}(\text{id}_{\Delta^n}).$$

By the naturality condition, the diagram

$$\begin{array}{ccc} S_n(\Delta^n) & \xrightarrow{D_{\Delta^n}} & S_{n+1}(\Delta^n \times I) \\ T_{\#} \downarrow & & \downarrow (T \times \text{id})_{\#} \\ S_n(X) & \xrightarrow{D_X} & S_{n+1}(X \times I) \end{array}$$

must commute. Now, $\text{id}_{\Delta^n} \in S_n(\Delta^n)$, and $T_{\#}(\text{id}_{\Delta^n}) = T \in S_n(X)$. By the commutativity,

$$D_X(T) = (D_X \circ T_{\#})(\text{id}_{\Delta^n}) = (T \times \text{id})_{\#} D_{\Delta^n}(\text{id}_{\Delta^n}).$$

Define

$$D_{\Delta^n}(\text{id}_{\Delta^n}) = b.$$

Then

$$D_X(T) = (T \times \text{id})_{\#}(b).$$

Thus we obtain a homomorphism

$$D_X: S_n(X) \rightarrow S_{n+1}(X), \quad \sum n_i T_i \mapsto \sum n_i D_X(T_i).$$

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & S_{n+2}(X) & \xrightarrow{\partial_{n+2}} & S_{n+1}(X) & \xrightarrow{\partial_{n+1}} & S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \longrightarrow \cdots \\
& & \downarrow & \swarrow D_{n+1} & \downarrow & \swarrow D_n & \downarrow \swarrow D_{n-1} \\
\cdots & \longrightarrow & S_{n+2}(X \times I) & \xrightarrow{\partial'_{n+2}} & S_{n+1}(X \times I) & \xrightarrow{\partial'_{n+1}} & S_n(X \times I) \xrightarrow{\partial'_n} S_{n-1}(X \times I) \longrightarrow \cdots
\end{array}$$

$\downarrow (i_0)_\#, (i_1)_\#$

By the induction assumption,

$$\partial_{j+1}(D_X)_j + (D_X)_{j-1}\partial_j = (i_1)_\# - (i_0)_\#,$$

for $j+1 \leq n$, i.e., for $j < n$. Let now $j = n$. Then

$$\begin{aligned}
\partial D_X(T) + D_X(\partial T) &= \partial(T \times \text{id})_\#(b) + D_X \partial(T_\#(\text{id}_{\Delta^n})) \\
&= (T \times \text{id})_\# \underbrace{\partial b}_c + D_X T_\# \partial(\text{id}_{\Delta^n}) \\
&= (T \times \text{id})_\#(c) + (T \times \text{id})_\# D_{\Delta^n}(\partial \text{id}_{\Delta^n}) \\
&= (T \times \text{id})_\#((i_1)_\#(\text{id}_{\Delta^n}) - (i_0)_\#(\text{id}_{\Delta^n}) - D_{\Delta^n}(\partial \text{id}_{\Delta^n})) \\
&\quad + (T \times \text{id})_\# D_{\Delta^n}(\partial \text{id}_{\Delta^n}) \text{ (the diagram } \spadesuit \text{ commutes)} \\
&= (T \times \text{id})_\#((i_1)_\#(\text{id}_{\Delta^n})) - (T \times \text{id})_\#((i_0)_\#(\text{id}_{\Delta^n})) \\
&= (i_1 T)_\#(\text{id}_{\Delta^n}) - (i_0 T)_\#(\text{id}_{\Delta^n}) \\
&= (i_1)_\#(T) - (i_0)_\#(T) = ((i_1)_\# - (i_0)_\#)(T).
\end{aligned}$$

Thus

$$\partial D_X + D_X \partial = (i_1)_\# - (i_0)_\#.$$

Thus $D_X: S_n(X) \rightarrow S_{n+1}(X \times I)$ is a chain homotopy between $(i_1)_\#$ and $(i_0)_\#$ up to dimension n . We still have to check that the naturality condition holds:

Let $h: X \rightarrow X'$. Then

$$\begin{aligned}
(h \times \text{id})_\#(D_X T) &= (h \times \text{id})_\#(T \times \text{id})_\#(b) \\
&= (hT \times \text{id})_\#(b) \\
&= D_{X'}(hT) \\
&= D_{X'} h_\#(T),
\end{aligned}$$

i.e.,

$$(h \times \text{id})_\# D_X = D_{X'} h_\#,$$

which means that the diagram

$$\begin{array}{ccc}
S_n(X) & \xrightarrow{D_X} & S_{n+1}(X \times I) \\
h_\# \downarrow & & \downarrow (h \times \text{id})_\# \\
S_n(X') & \xrightarrow{D_{X'}} & S_{n+1}(X' \times I)
\end{array}$$

commutes. Thus we are done with the induction step.

We have constructed a chain homotopy $D_X: S_*(X) \rightarrow S_*(X \times I)$ between $(i_1)_\#$ and $(i_0)_\#$, satisfying the naturality condition.

The relative case: Let $A \subset X$ and let $i: A \hookrightarrow X$ be the inclusion. We now have chain homotopies D_A and D_X such that the diagram

$$\begin{array}{ccc} S_n(A) & \xrightarrow{D_A} & S_{n+1}(A \times I) \\ i_\# \downarrow & & \downarrow (i \times \text{id})_\# \\ S_n(X) & \xrightarrow{D_X} & S_{n+1}(X \times I) \end{array}$$

commutes. Thus $D_X|_{S_n(A)} = D_A$, for every n . Therefore, D_X induces a chain homotopy

$$D(X, A): \underbrace{S_n(X)/S_n(A)}_{=S_n(X, A)} \rightarrow \underbrace{S_{n+1}(X \times I)/S_{n+1}(A \times I)}_{=S_{n+1}(X \times I, A \times I)}$$

from $(i_1)_\#$ to $(i_1)_\#$, where

$$(i_0)_\#, (i_1)_\#: S_*(X, A) \rightarrow S_*(X \times I, A \times I)$$

are chain maps. Assume $f, g: (X, A) \rightarrow (Y, B)$ are homotopic, $F: f \simeq g$. Then

$$F_\# D(X, A): S_*(X, A) \rightarrow S_*(Y, B)$$

is a chain homotopy from $g_\#$ to $f_\#$, where

$$F_\#: S_*(X \times I, A \times I) \rightarrow S_*(Y, B)$$

is the chain map induced by $F: (X \times I, A \times I) \rightarrow (Y, B)$. □

19. REDUCED HOMOLOGY

Let X be a topological space. The singular chain complex of X is

$$\cdots \rightarrow S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \xrightarrow{\partial_{n-1}} S_{n-2}(X) \rightarrow \cdots \rightarrow S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\partial_0} 0.$$

To define the *reduced homology* $\tilde{H}_*(X)$ we change $S_{-1}(X)$ and ∂_0 as follows:

$$\cdots \rightarrow S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\partial'_0} \mathbb{Z}, \quad (*)$$

where

$$\partial'_0\left(\sum_x n_x x\right) = \sum_x n_x \quad (\text{finite sums}).$$

Let $T: \Delta^1 \rightarrow X$. Then

$$\partial_1 T = T \circ e^0 - T \circ e^1 \in S_0(X),$$

and

$$\partial'_0(\partial_1 T) = 1 + (-1) = 0 \implies \partial'_0 \circ \partial_1 = 0.$$

Thus $(*)$ is a chain complex. The reduced homology groups $\tilde{H}_*(X)$ of X are

$$\tilde{H}_n(X) = H_n(X), \text{ for } n > 0, \text{ and } \tilde{H}_0(X) = \ker \partial'_0 / \text{im } \partial_1.$$

Remark 19.1. Notice that we already had the homomorphism $\partial'_0: S_0(X) \rightarrow \mathbb{Z}$ in Proposition 14.3, where we proved that $H_0(X) \cong \mathbb{Z}$ for a path-connected space X . There we used the notation η for ∂'_0 and proved that $\ker \eta = \text{im}(\partial_1(S_1)) = B_0(X)$. Thus $\tilde{H}_0(X) = \ker \partial'_0 / \text{im} \partial_1 = 0$, for path-connected X .

We will show that $\tilde{H}_n(X) \cong H_n(X, \{x_0\})$, where $x_0 \in X$, for any topological space X . Notation $H_n(X, x_0)$ will be used for $H_n(X, \{x_0\})$. First we will have to do some preliminary work.

Let X be a topological space and let A be a subspace of X . We have the singular chain complexes $S_*(X)$ and $S_*(A)$, and the quotient complex $S_*(X)/S_*(A)$. We defined the relative homology groups:

$$H_n(X, A) = H_n(S_*(X)/S_*(A)).$$

There is another way to consider these groups: Let

$$\partial_n: S_n(X) \rightarrow S_{n-1}(X)$$

be the boundary map, for all n . The ∂_n induce boundary homomorphisms $\bar{\partial}_n$ for $S_*(X)/S_*(A)$,

$$\bar{\partial}_n: S_n(X)/S_n(A) \rightarrow S_{n-1}(X)/S_{n-1}(A), \quad \gamma + S_n(A) \mapsto \partial_n \gamma + S_{n-1}(A).$$

Then

$$\ker \bar{\partial}_n = \{\gamma + S_n(A) \mid \partial_n \gamma \in S_{n-1}(A)\}$$

and

$$\text{im} \bar{\partial}_{n+1} = \{\gamma + S_n(A) \mid \gamma \in \text{im} \partial_{n+1} = B_n(X)\}.$$

Definition 19.2. The group of *relative n -cycles mod A* is

$$Z_n(X, A) = \{\gamma \in S_n(X) \mid \partial_n \gamma \in S_{n-1}(A)\},$$

and the group of *relative n -boundaries mod A* is

$$\begin{aligned} B_n(X, A) &= \{\gamma \in S_n(X) \mid \gamma - \gamma' \in B_n(X) \text{ for some } \gamma' \in S_n(A)\} \\ &= B_n(X) + S_n(A). \end{aligned}$$

Then

$$S_n(A) \subset B_n(X, A) \subset Z_n(X, A) \subset S_n(X).$$

Theorem 19.3. For all $n \geq 0$,

$$H_n(X, A) \cong Z_n(X, A)/B_n(X, A).$$

Proof. By definition,

$$H_n(X, A) = \ker \bar{\partial}_n / \text{im} \bar{\partial}_{n+1},$$

where

$$\ker \bar{\partial}_n = Z_n(X, A)/S_n(A)$$

and

$$\text{im} \bar{\partial}_{n+1} = B_n(X, A)/S_n(A).$$

Thus

$$\begin{aligned} H_n(X, A) &= (Z_n(X, A)/S_n(A))/(B_n(X, A)/S_n(A)) \\ &\cong Z_n(X, A)/B_n(X, A). \end{aligned}$$

□

Theorem 19.4. *Let X be a path-connected topological space, and let $A \subset X$, $A \neq \emptyset$. Then $H_0(X, A) = 0$.*

Proof. Since $A \neq \emptyset$, there is some $x_0 \in A$. Let

$$\gamma = \sum_i n_i x_i \in Z_0(X, A) = S_0(X),$$

where we identify $T_i: \Delta^0 \rightarrow X$ with $x_i = T_i(1)$. Since X is path-connected, there are continuous maps

$$\sigma_i: \Delta^1 \rightarrow X,$$

where

$$\sigma_i(e_0) = x_0, \text{ and } \sigma_i(e_1) = x_i$$

for every i . (Recall that $e_0 = (1, 0)$ and $e_1 = (0, 1)$.) Then $\sum_i n_i \sigma_i \in S_1(X)$, and

$$\begin{aligned} \partial_1(\sum_i n_i \sigma_i) &= \sum_i n_i \partial_1 \sigma_i = \sum_i n_i (x_i - x_0) \\ &= \sum_i n_i x_i - (\sum_i n_i) x_0 = \gamma - \underbrace{(\sum_i n_i) x_0}_{\gamma'}. \end{aligned}$$

Here

$$\gamma' = (\sum_i n_i) x_0 \in S_0(A).$$

Thus

$$\gamma - \gamma' = \partial_1(\sum_i n_i \sigma_i) \in B_0(X),$$

and hence $\gamma \in B_0(X, A)$. It follows that

$$B_0(X, A) = Z_0(X, A).$$

Hence

$$H_0(X, A) = 0.$$

□

Theorem 19.5. *Let X be a topological space, and let $\{X_\lambda \mid \lambda \in \Lambda\}$ be the family of path components of X . Then, for every $n \geq 0$,*

$$H_n(X, A) \cong \sum_\lambda H_n(X_\lambda, A \cap X_\lambda).$$

Proof. First show that for a direct sum of chain complexes, $S_* = \sum_{\lambda} S_*^{\lambda}$,

$$H_n\left(\sum_{\lambda} S_*^{\lambda}\right) \cong \sum_{\lambda} H_n(S_*^{\lambda}).$$

Then write

$$S_*(X) = \sum_{\lambda} S_*(X_{\lambda}) \quad \text{and} \quad S_*(A) = \sum_{\lambda} S_*(A \cap X_{\lambda}).$$

Thus

$$\begin{aligned} S_*(X, A) &= S_*(X)/S_*(A) \\ &= \sum_{\lambda} S_*(X_{\lambda}) / \sum_{\lambda} S_*(A \cap X_{\lambda}) \\ &\cong \sum_{\lambda} (S_*(X_{\lambda})/S_*(A \cap X_{\lambda})), \end{aligned}$$

which implies that for every $n \geq 0$,

$$\begin{aligned} H_n(X, A) &= H_n(S_*(X, A)) \\ &\cong H_n\left(\sum_{\lambda} (S_*(X_{\lambda})/S_*(A \cap X_{\lambda}))\right) \\ &\cong \sum_{\lambda} H_n(S_*(X_{\lambda})/S_*(A \cap X_{\lambda})) \\ &= \sum_{\lambda} H_n(X_{\lambda}, A \cap X_{\lambda}). \end{aligned}$$

□

The *rank* of a free abelian group means the cardinality of the basis of the group. Any two bases of a free abelian group have the same cardinality.

Corollary 19.6. *For every topological space X and for every subspace A of X , the group $H_0(X, A)$ is free abelian and*

$$\text{rank} H_0(X, A) = \text{card}\{\lambda \in \Lambda \mid A \cap X_{\lambda} = \emptyset\},$$

where $\{X_{\lambda} \mid \lambda \in \Lambda\}$ is the family of the path components of X .

Proof. By Theorem 19.5,

$$H_0(X, A) \cong \sum_{\lambda} H_0(X_{\lambda}, A \cap X_{\lambda}).$$

If $A \cap X_{\lambda} = \emptyset$, then

$$H_0(X_{\lambda}, A \cap X_{\lambda}) = H_0(X_{\lambda}, \emptyset) \cong H_0(X_{\lambda}) \cong \mathbb{Z},$$

since X_{λ} is path-connected. If $A \cap X_{\lambda} \neq \emptyset$, then Theorem 19.4 implies that $H_0(X_{\lambda}, A \cap X_{\lambda}) = 0$. □

Corollary 19.7. *Let X be a topological space, and let $x_0 \in X$. Then $H_0(X, x_0)$ is a free abelian group of rank r , where X has exactly $r + 1$ path components. (Here r may equal ∞ , in which case $\infty + 1 = \infty$.)*

Proof. Let X_{λ_0} be the unique path component of X containing x_0 . Then $\{x_0\} \cap X_\lambda = \emptyset$, for every $\lambda \neq \lambda_0$. Thus $H_0(X_\lambda, \{x_0\} \cap X_\lambda) \cong \mathbb{Z}$ for every $\lambda \neq \lambda_0$ and $H_0(X_{\lambda_0}, x_0) = 0$. $\square$

Theorem 19.8. *Let X be a topological space with basepoint x_0 . Then*

$$H_n(X, x_0) \cong H_n(X),$$

for all $n \geq 1$.

Proof. The inclusion $\{x_0\} \hookrightarrow X$ induces the exact sequence in homology

$$\cdots \longrightarrow H_n(\{x_0\}) \longrightarrow H_n(X) \xrightarrow{g} H_n(X, x_0) \rightarrow H_{n-1}(\{x_0\}) \longrightarrow \cdots$$

Let $n \geq 2$. Then $n - 1 \geq 1$. The Dimension axiom for homology implies that $H_n(\{x_0\}) = 0 = H_{n-1}(\{x_0\})$. It follows that g is an isomorphism. Thus $H_n(X) \cong H_n(X, x_0)$, for $n \geq 2$.

Let then $n = 1$. Consider the tail of the exact homology sequence:

$$\underbrace{H_1(\{x_0\})}_0 \longrightarrow H_1(X) \xrightarrow{g} H_1(X, x_0) \xrightarrow{f} H_0(\{x_0\}) \xrightarrow{h} H_0(X) \xrightarrow{k} H_0(X, x_0) \longrightarrow 0.$$

Then $H_1(\{x_0\}) = 0$ implies that $\ker g = 0$, and it follows that g is an injection.

Assume h is an injection. Then $\operatorname{im} f = \ker h = 0$, and it follows that $\operatorname{im} g = \ker f = H_1(X, x_0)$. Therefore, to show that g is surjective, it suffices to show that h is injective. Since $H_0(\{x_0\}) \cong \mathbb{Z}$ and $H_0(X)$ is free abelian, it follows that h is injective if it is not the zero homomorphism, i.e., if $\operatorname{im} h \neq 0$. Thus it suffices to check that $\operatorname{im} h = \ker k \neq 0$.

The homomorphism $k: H_0(X) \rightarrow H_0(X, x_0)$ is induced by the inclusion $X \hookrightarrow (X, x_0)$. Now, $Z_0(X) = S_0(X)$, so that $H_0(X) = S_0(X)/B_0(X)$. By Theorem 19.3,

$$H_0(X, x_0) = Z_0(X, x_0)/B_0(X, x_0),$$

where

$$Z_0(X, x_0) = S_0(X) \quad \text{and} \quad B_0(X, x_0) = B_0(X) + S_0(x_0).$$

Thus

$$\begin{aligned} k: S_0(X)/B_0(X) &\rightarrow S_0(X)/(B_0(X) + S_0(x_0)), \\ \gamma + B_0(X) &\mapsto \gamma + B_0(X) + S_0(x_0). \end{aligned}$$

Therefore,

$$\ker k = (B_0(X) + S_0(x_0))/B_0(X).$$

But, by the proof of Proposition 14.3,

$$B_0(X) = \left\{ \sum m_x x \in S_0(X) \mid \sum m_x = 0 \right\}.$$

Thus $\ker k \neq 0$. $\square$

Theorem 19.9. *Let X be a topological space and let $x_0 \in X$. For all $n \geq 0$,*

$$\tilde{H}_n(X) \cong H_n(X, x_0).$$

Proof. For $n \geq 1$,

$$\tilde{H}_n(X) = H_n(X) \cong H_n(X, x_0),$$

by Theorem 19.8.

Let then $n = 0$. The tail of the reduced singular chain complex of X is

$$\cdots \longrightarrow S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\partial'_0} S_{-1}(X) = \mathbb{Z},$$

where $\partial'_0(\sum_x n_x x) = \sum_x n_x$. This induces the short exact sequence

$$0 \longrightarrow \ker \partial'_0 \longrightarrow S_0(X) \xrightarrow{\partial'_0} \mathbb{Z} \longrightarrow 0,$$

since ∂'_0 is a surjection. Let $\alpha \in S_0(X)$ be such that $\partial'_0(\alpha) = 1$.

Claim: Let $\langle \alpha \rangle$ denote the infinite cyclic group generated by α . Then

$$S_0(X) = \ker \partial'_0 \oplus \langle \alpha \rangle.$$

Proof of the claim: Let $\gamma \in S_0(X)$, and let $\partial'_0(\gamma) = k$. Then

$$\gamma = (\gamma - k\alpha) + k\alpha \in \ker \partial'_0 + \langle \alpha \rangle.$$

Assume $x \in \ker \partial'_0 \cap \langle \alpha \rangle$. Then $x = m\alpha$, for some $m \in \mathbb{Z}$ and $\partial'_0(x) = 0$ and $\partial'_0(x) = \partial'_0(m\alpha) = m\partial'_0(\alpha) = m$. Thus $m = 0$, and consequently $x = 0$. This proves the claim.

Since $\partial'_0 \partial_1 = 0$, it follows that $B_0(X) = \text{im } \partial_1 \subset \ker \partial'_0$. Since $S_0(X) = Z_0(X)$, it follows that

$$\begin{aligned} H_0(X) &= S_0(X)/B_0(X) \\ &= (\ker \partial'_0 \oplus \langle \alpha \rangle)/B_0(X) \\ &\cong (\ker \partial'_0/B_0(X)) \oplus \mathbb{Z} \\ &= \tilde{H}_0(X) \oplus \mathbb{Z}. \end{aligned}$$

It now follows from Corollary 19.6, that $\tilde{H}_0(X) \cong H_0(X, x_0)$. □

20. EXCISION AND MAYER-VIETORIS SEQUENCES

In this section we discuss the Excision axiom for singular homology. Later we will prove that singular homology satisfies the Excision axiom. Recall that if X is a topological space and $A \subset X$, then the interior of A is denoted by $\mathring{A}$. There are two equivalent ways to formulate the Excision axiom:

Theorem 20.1. (*Excision 1.*) *Let X be a topological space and let $U \subset A \subset X$ be subspaces with $\overline{U} \subset \mathring{A}$. Then the inclusion*

$$i: (X \setminus U, A \setminus U) \hookrightarrow (X, A)$$

induces isomorphisms

$$i_*: H_n(X \setminus U, A \setminus U) \rightarrow H_n(X, A),$$

for every n .

Theorem 20.2. (*Excision 2.*) Let X be a topological space and let X_1 and X_2 be subspaces of X with $X = \mathring{X}_1 \cup \mathring{X}_2$. Then the inclusion

$$j: (X_1, X_1 \cap X_2) \hookrightarrow (X_1 \cup X_2, X_2) = (X, X_2)$$

induces isomorphisms

$$j_*: H_n(X_1, X_1 \cap X_2) \rightarrow H_n(X, X_2),$$

for every n .

Theorem 20.3. *Excision 1 is equivalent with Excision 2.*

Proof. Assume first that Excision 1 holds. Let X_1 and X_2 be subspaces of X with $X = \mathring{X}_1 \cup \mathring{X}_2$. Let $A = X_2$ and $U = X \setminus X_1$.

Now,

$$\begin{aligned} \mathring{X}_1 \subset X_1 &\implies X \setminus X_1 \subset X \setminus \mathring{X}_1 \leftarrow \text{closed set} \\ &\implies \overline{U} = \overline{X \setminus X_1} \subset X \setminus \mathring{X}_1, \end{aligned}$$

and

$$\begin{aligned} X \setminus \mathring{X}_1 &= (\mathring{X}_1 \cup \mathring{X}_2) \setminus \mathring{X}_1 = \mathring{X}_2 \setminus \mathring{X}_1 \\ &\subset \mathring{X}_2 = \mathring{A}. \end{aligned}$$

Thus $\overline{U} \subset \mathring{A}$. Also,

$$X \setminus U = X \setminus (X \setminus X_1) = X_1$$

and

$$A \setminus U = X_2 \setminus (X \setminus X_1) = X_1 \cap X_2.$$

Thus

$$(X \setminus U, A \setminus U) = (X_1, X_1 \cap X_2) \text{ and } (X, A) = (X, X_2).$$

By Excision 1, the inclusion

$$(X_1, X_1 \cap X_2) = (X \setminus U, A \setminus U) \hookrightarrow (X, A) = (X, X_2)$$

induces isomorphisms

$$H_n(X_1, X_1 \cap X_2) \rightarrow H_n(X, X_2),$$

for every n .

Assume then that Excision 2 holds. Let U and A be subspaces of X such that $\overline{U} \subset \mathring{A}$. Define

$$X_1 = X \setminus U \text{ and } X_2 = A.$$

Since $U \subset \overline{U} \subset \mathring{A}$, it follows that

$$X \setminus \mathring{A} \subset X \setminus \overline{U} \subset X \setminus U.$$

Since $X \setminus \overline{U}$ is open,

$$X \setminus \mathring{A} \subset X \setminus \overline{U} = (X \setminus \overline{U})^\circ.$$

Thus

$$\begin{aligned} X &= (X \setminus \mathring{A}) \cup \mathring{A} \subset (X \setminus \overline{U})^\circ \cup \mathring{A} \\ &\subset (X \setminus U)^\circ \cup \mathring{A} = \mathring{X}_1 \cup \mathring{X}_2. \end{aligned}$$

Then

$$(X_1, X_1 \cap X_2) = (X \setminus U, A \setminus U) \text{ and } (X, X_2) = (X, A)$$

and the inclusion

$$(X \setminus U, A \setminus U) = (X_1, X_1 \cap X_2) \hookrightarrow (X, X_2) = (X, A)$$

induces isomorphisms

$$H_n(X \setminus U, A \setminus U) \rightarrow H_n(X, A),$$

for every n . □

Lemma 20.4. (*Barratt-Whitehead*) Consider the following commutative diagram of abelian groups and homomorphisms

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & A_n & \xrightarrow{i_n} & B_n & \xrightarrow{p_n} & C_n & \xrightarrow{d_n} & A_{n-1} & \longrightarrow & \cdots \\ & & \downarrow f_n & & \downarrow g_n & & \downarrow h_n & & \downarrow f_{n-1} & & \\ \cdots & \longrightarrow & A'_n & \xrightarrow{j_n} & B'_n & \xrightarrow{q_n} & C'_n & \xrightarrow{\Delta_n} & A'_{n-1} & \longrightarrow & \cdots \end{array}$$

where the rows are exact sequences and the homomorphisms h_n are isomorphisms. Then there is an exact sequence

$$\cdots \longrightarrow A_n \xrightarrow{(i_n, f_n)} B_n \oplus A'_n \xrightarrow{g_n - j_n} B'_n \xrightarrow{d_n h_n^{-1} q_n} A_{n-1} \longrightarrow \cdots,$$

where

$$(i_n, f_n)(a_n) = (i_n(a_n), f_n(a_n)) \text{ and } (g_n - j_n)(b_n, a'_n) = g_n(b_n) - j_n(a'_n).$$

Proof. Exercise. □

Theorem 20.5. (*Mayer - Vietoris*) Let X be a topological space and let X_1, X_2 be subspaces of X with $X = \overset{\circ}{X}_1 \cup \overset{\circ}{X}_2$. Then there is an exact sequence

$$\cdots \longrightarrow H_n(X_1 \cap X_2) \xrightarrow{(i_{1,*}, i_{2,*})} H_n(X_1) \oplus H_n(X_2) \xrightarrow{g_* - j_*} H_n(X) \xrightarrow{D} H_{n-1}(X_1 \cap X_2) \longrightarrow \cdots,$$

where

- (1) $i_1: X_1 \cap X_2 \hookrightarrow X_1$,
- (2) $i_2: X_1 \cap X_2 \hookrightarrow X_2$,
- (3) $g: X_1 \hookrightarrow X$,
- (4) $j: X_2 \hookrightarrow X$,
- (5) $q: (X, \emptyset) \hookrightarrow (X, X_2)$, and
- (6) $h: (X_1, X_1 \cap X_2) \hookrightarrow (X, X_2)$

are inclusions, $d: H_n(X_1, X_1 \cap X_2) \rightarrow H_{n-1}(X_1 \cap X_2)$ is a connecting homomorphism, and $D = dh_*^{-1}q_*$.

Proof. The following diagram commutes, all maps are inclusions:

$$\begin{array}{ccccc}
 (X_1 \cap X_2, \emptyset) & \xrightarrow{i_1} & (X_1, \emptyset) & \xrightarrow{p} & (X_1, X_1 \cap X_2) \\
 \downarrow i_2 & & \downarrow g & & \downarrow h \\
 (X_2, \emptyset) & \xrightarrow{j} & (X, \emptyset) & \xrightarrow{q} & (X, X_2)
 \end{array}$$

This diagram induces a commutative diagram of chain complexes:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & S_*(X_1 \cap X_2) & \xrightarrow{(i_1)_\#} & S_*(X_1) & \xrightarrow{p_\#} & S_*(X_1, X_1 \cap X_2) \longrightarrow 0 \\
 & & \downarrow (i_2)_\# & & \downarrow g_\# & & \downarrow h_\# \\
 0 & \longrightarrow & S_*(X_2) & \xrightarrow{j_\#} & S_*(X) & \xrightarrow{q_\#} & S_*(X, X_2) \longrightarrow 0,
 \end{array}$$

where all the horizontal lines are exact. By Corollary 15.13, the diagram

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & H_n(X_1 \cap X_2) & \xrightarrow{(i_1)_*} & H_n(X_1) & \xrightarrow{p_*} & H_n(X_1, X_1 \cap X_2) \xrightarrow{d} H_{n-1}(X_1 \cap X_2) \longrightarrow \cdots \\
 & & \downarrow (i_2)_* & & \downarrow g_* & & \downarrow h_* & & \downarrow (i_2)_* \\
 \cdots & \longrightarrow & H_n(X_2) & \xrightarrow{j_*} & H_n(X) & \xrightarrow{q_*} & H_n(X, X_2) \xrightarrow{\Delta} H_{n-1}(X_2) \longrightarrow \cdots
 \end{array}$$

commutes and the horizontal lines are exact. Here d and Δ are connecting homomorphisms. The homomorphisms h_* are isomorphisms by Excision 2. By Lemma 20.4, there is an exact sequence

$$\cdots \longrightarrow H_n(X_1 \cap X_2) \xrightarrow{(i_{1,*}, i_{2,*})} H_n(X_1) \oplus H_n(X_2) \xrightarrow{g_* - j_*} H_n(X) \xrightarrow{D} H_{n-1}(X_1 \cap X_2) \longrightarrow \cdots$$

□

There is a corresponding result for reduced homology:

Theorem 20.6. (*Mayer - Vietoris theorem for reduced homology*) Let X be a topological space and let X_1, X_2 be subspaces of X with $X = \overset{\circ}{X}_1 \cup \overset{\circ}{X}_2$, and $X_1 \cap X_2 \neq \emptyset$. Then there is an exact sequence

$$\cdots \longrightarrow \tilde{H}_n(X_1 \cap X_2) \xrightarrow{(i_{1,*}, i_{2,*})} \tilde{H}_n(X_1) \oplus \tilde{H}_n(X_2) \xrightarrow{g_* - j_*} \tilde{H}_n(X) \xrightarrow{D} \tilde{H}_{n-1}(X_1 \cap X_2) \longrightarrow \cdots,$$

that ends

$$\cdots \longrightarrow \tilde{H}_0(X_1) \oplus \tilde{H}_0(X_2) \longrightarrow \tilde{H}_0(X) \longrightarrow 0.$$

The induced maps are as in Theorem 20.5.

Proof. Let $x_0 \in X_1 \cap X_2$. We have the following commutative diagram of inclusions of pairs

$$\begin{array}{ccccc} (X_1 \cap X_2, x_0) & \xrightarrow{i_1} & (X_1, x_0) & \xrightarrow{p} & (X_1, X_1 \cap X_2) \\ \downarrow i_2 & & \downarrow g & & \downarrow h \\ (X_2, x_0) & \xrightarrow{j} & (X, x_0) & \xrightarrow{q} & (X, X_2). \end{array}$$

Then the following diagram with exact rows commutes:

$$\begin{array}{ccccccc} \cdots \longrightarrow & H_n(X_1 \cap X_2, x_0) & \xrightarrow{(i_1)_*} & H_n(X_1, x_0) & \xrightarrow{p_*} & H_n(X_1, X_1 \cap X_2) & \xrightarrow{d} H_{n-1}(X_1 \cap X_2, x_0) \longrightarrow \cdots \\ & \downarrow (i_2)_* & & \downarrow g_* & & \downarrow h_* & \downarrow (i_2)_* \\ \cdots \longrightarrow & H_n(X_2, x_0) & \xrightarrow{j_*} & H_n(X, x_0) & \xrightarrow{q_*} & H_n(X, X_2) & \xrightarrow{\Delta} H_{n-1}(X_2, x_0) \longrightarrow \cdots \end{array}$$

By excision, h_* is an isomorphism. By the Barratt - Whitehead lemma (Lemma 20.4), there is an exact sequence

$$\begin{aligned} \cdots \longrightarrow H_n(X_1 \cap X_2, x_0) & \xrightarrow{(i_{1,*}, i_{2,*})} H_n(X_1, x_0) \oplus H_n(X_2, x_0) \xrightarrow{g_* - j_*} H_n(X, x_0) \xrightarrow{D} \\ & H_{n-1}(X_1 \cap X_2, x_0) \longrightarrow \cdots \end{aligned}$$

Then, by Theorem 19.9, there is an exact sequence

$$\cdots \longrightarrow \tilde{H}_n(X_1 \cap X_2) \longrightarrow \tilde{H}_n(X_1) \oplus \tilde{H}_n(X_2) \longrightarrow \tilde{H}_n(X) \longrightarrow \tilde{H}_{n-1}(X_1 \cap X_2) \longrightarrow \cdots$$

□

21. APPLICATIONS OF EXCISION AND MAYER - VIETORIS SEQUENCES

Before proving that singular homology satisfies excision we introduce some applications.

Theorem 21.1. *Let $\mathbb{S}^n$ be the n -sphere, $n \geq 0$. Then*

$$H_p(\mathbb{S}^0) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & \text{if } p = 0, \\ 0, & \text{if } p > 0. \end{cases}$$

If $n > 0$, then

$$H_p(\mathbb{S}^n) = \begin{cases} \mathbb{Z}, & \text{if } p = 0 \text{ or } p = n, \\ 0, & \text{otherwise.} \end{cases}$$

Using reduced homology these claims can be combined: For every $n \geq 0$,

$$\tilde{H}_p(\mathbb{S}^n) = \begin{cases} \mathbb{Z}, & \text{if } p = n, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The proof is by induction on n :

Let $n = 0$. Then $\mathbb{S}^0 = \{-1, 1\}$. By Theorem 19.9 and Corollary 19.7,

$$\tilde{H}_0(\mathbb{S}^0) \cong H_0(\mathbb{S}^0, 1) \cong \mathbb{Z}.$$

For $p > 0$, Theorems 19.8 and 19.9 imply that

$$\tilde{H}_p(\mathbb{S}^0) \cong H_p(\mathbb{S}^0, 1) \cong H_p(\mathbb{S}^0) = 0.$$

Thus the claim holds for $n = 0$.

Assume then $n > 0$. Let a and b be the north and the south poles of $\mathbb{S}^n$, respectively. Let $X_1 = \mathbb{S}^n \setminus \{a\}$ and $X_2 = \mathbb{S}^n \setminus \{b\}$. Then $\mathring{X}_1 = X_1$, $\mathring{X}_2 = X_2$ and $\mathbb{S}^n = \mathring{X}_1 \cup \mathring{X}_2$, X_1 and X_2 are contractible (homeomorphic to $\mathbb{R}^n$). The intersection $X_1 \cap X_2 = \mathbb{S}^n \setminus \{a, b\}$ has the same homotopy type as the equator $\mathbb{S}^{n-1}$. Mayer - Vietoris sequence for reduced homology yields the long exact sequence

$$\begin{aligned} \cdots \longrightarrow \tilde{H}_p(X_1) \oplus \tilde{H}_p(X_2) \longrightarrow \tilde{H}_p(\mathbb{S}^n) \longrightarrow \tilde{H}_{p-1}(X_1 \cap X_2) \longrightarrow \\ \tilde{H}_{p-1}(X_1) \oplus \tilde{H}_{p-1}(X_2) \longrightarrow \cdots \end{aligned}$$

where all the reduced homology groups of X_1 and X_2 are trivial. Thus

$$\begin{aligned} \tilde{H}_p(\mathbb{S}^n) &\cong \tilde{H}_{p-1}(X_1 \cap X_2) \\ &\cong \tilde{H}_{p-1}(\mathbb{S}^{n-1}) \\ &\stackrel{\simeq}{=} \begin{cases} \mathbb{Z}, & \text{if } p-1 = n-1, \\ 0, & \text{if } p-1 \neq n-1, \end{cases} \\ &= \begin{cases} \mathbb{Z}, & \text{if } p = n, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Induction is being used at $\simeq$. □

Corollary 21.2. *Let $n \geq 0$. Then $\mathbb{S}^n$ is not a retract of $\mathbb{D}^{n+1}$.*

Proof. Assume $\mathbb{S}^n$ is a retract of $\mathbb{D}^{n+1}$. Then there is a continuous map $r: \mathbb{D}^{n+1} \rightarrow \mathbb{S}^n$ such that $r \circ i: \mathbb{S}^n \rightarrow \mathbb{S}^n$ is the identity, where $i: \mathbb{S}^n \hookrightarrow \mathbb{D}^{n+1}$ is the inclusion. Thus there are homomorphisms

$$H_n(\mathbb{S}^n) \xrightarrow{i_*} H_n(\mathbb{D}^{n+1}) \xrightarrow{r_*} H_n(\mathbb{S}^n)$$

where the composition is

$$r_* \circ i_* = (r \circ i)_* = \text{id}_* = \text{id}: H_n(\mathbb{S}^n) \rightarrow H_n(\mathbb{S}^n).$$

This is impossible, since $H_n(\mathbb{S}^n) \cong \mathbb{Z}$ and $H_n(\mathbb{D}^{n+1}) = 0$. □

Corollary 21.3. *(Brouwer's fixed point theorem) Let $f: \mathbb{D}^n \rightarrow \mathbb{D}^n$ be continuous. Then there is $x \in \mathbb{D}^n$ with $f(x) = x$.*

Proof. Assume $f(x) \neq x$, for all $x \in \mathbb{D}^n$. Then the points x and $f(x)$ determine a line. Let $g: \mathbb{D}^n \rightarrow \mathbb{S}^{n-1}$ be the function that assigns to x the point where the ray from $f(x)$ to x intersects $\mathbb{S}^{n-1}$. Then g is continuous and $g(x) = x$, for every $x \in \mathbb{S}^{n-1}$. Thus g is a retraction, which contradicts Corollary 21.2. □

Corollary 21.4. *If $m \neq n$, then $\mathbb{S}^m$ and $\mathbb{S}^n$ do not have the same homotopy type. In particular, they are not homeomorphic.*

Proof. A homotopy equivalence would induce an isomorphism $H_p(\mathbb{S}^m) \cong H_p(\mathbb{S}^n)$, for all $p \geq 0$. □

Corollary 21.5. *If $m \neq n$, then $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic.*

Proof. Assume $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a homeomorphism. Then choose $x_0 \in \mathbb{R}^m$. The restriction

$$f|: \mathbb{R}^m \setminus \{x_0\} \rightarrow \mathbb{R}^n \setminus \{f(x_0)\}$$

is a homeomorphism. This is a contradiction, since $\mathbb{R}^m \setminus \{x_0\}$ has the same homotopy type as $\mathbb{S}^{m-1}$ while $\mathbb{R}^n \setminus \{f(x_0)\}$ has the same homotopy type as $\mathbb{S}^{n-1}$, but $\mathbb{S}^{m-1}$ and $\mathbb{S}^{n-1}$ do not have the same homotopy type. $\square$

Corollary 21.6. *If $n \geq 0$, then $\mathbb{S}^n$ is not contractible.*

Proof. If $\mathbb{S}^n$ were contractible, it would have the same homology groups as a point. $\square$

The following theorem says that the relative homology groups $H_n(X, A)$ can be considered as reduced homology groups of the quotient space X/A if A has a nice neighborhood in X .

Theorem 21.7. *Let X be a topological space and let A be a subspace of X . Assume A has a neighborhood V such that A is a strong deformation retract of V . Let*

$$q: (X, A) \rightarrow (X/A, A/A)$$

be the quotient map. Then, for all $n \geq 0$, q induces isomorphisms

$$q_*: H_n(X, A) \rightarrow H_n(X/A, A/A) \cong \tilde{H}_n(X/A).$$

Proof. We have the following commutative diagram, where the horizontal maps are induced by inclusions:

$$\begin{array}{ccccc} H_n(X, A) & \xrightarrow{f} & H_n(X, V) & \xleftarrow{g} & H_n(X \setminus A, V \setminus A) \\ \downarrow q_* & & \downarrow q_* & & \downarrow q_* \\ H_n(X/A, A/A) & \xrightarrow{h} & H_n(X/A, V/A) & \xleftarrow{l} & H_n((X/A) \setminus (A/A), (V/A) \setminus (A/A)) \end{array}$$

Let's consider the homomorphisms f . Since A is a strong deformation retract of V , it follows that the inclusion $i: A \hookrightarrow V$ is a homotopy equivalence. Thus $i_*: H_n(A) \rightarrow H_n(V)$ is an isomorphism, for every n . Consider the exact sequence

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(V) \xrightarrow{p} H_n(V, A) \xrightarrow{\Delta} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(V) \longrightarrow \cdots$$

Since i_* (on the right) is an isomorphism, it follows that $\text{im}(\Delta) = \ker(i_*) = 0$. Thus $\ker(\Delta) = H_n(V, A)$. Since i_* (on the left) is an isomorphism, it follows that $\ker(p) = \text{im}(i_*) = H_n(V)$. Thus $\text{im}(p) = 0$. Exactness of the sequence then implies that

$$0 = \text{im}(p) = \ker(\Delta) = H_n(V, A).$$

Consider the inclusions

$$(V, A) \hookrightarrow (X, A) \hookrightarrow (X, V).$$

They induce a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & S_*(A) & \longrightarrow & S_*(V) & \longrightarrow & S_*(V)/S_*(A) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & S_*(A) & \longrightarrow & S_*(X) & \longrightarrow & S_*(X)/S_*(A) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & S_*(V) & \longrightarrow & S_*(X) & \longrightarrow & S_*(X)/S_*(V) \longrightarrow 0
\end{array}$$

and a short exact sequence

$$0 \longrightarrow S_*(V)/S_*(A) \longrightarrow S_*(X)/S_*(A) \longrightarrow S_*(X)/S_*(V) \longrightarrow 0.$$

Hence there is a long exact sequence in homology:

$$\cdots \longrightarrow H_n(V, A) \xrightarrow{i_*} H_n(X, A) \xrightarrow{f} H_n(X, V) \xrightarrow{\Delta} H_{n-1}(V, A) \longrightarrow \cdots$$

where $H_n(V, A) = 0 = H_{n-1}(V, A)$. Then $\ker(f) = \text{im}(i_*) = 0$, and it follows that f is an injection. Also, $\text{im}(f) = \ker(\Delta) = H_n(X, V)$, and hence f is a surjection. Thus f is an isomorphism.

Since A is a strong deformation retract of V , it follows that A/A is a strong deformation retract of V/A . Thus, just as we proved that f is an isomorphism, we can show that h is an isomorphism.

It follows from excision that g and l are isomorphisms.

The quotient map $q: X \rightarrow X/A$ restricts to a homeomorphism on the complement of A . It follows that the q_* on the right is an isomorphism. Thus the q_* on the left is

$$(q_*)_{\text{left}} = h^{-1} \circ l \circ (q_*)_{\text{right}} \circ g^{-1} \circ f,$$

which is an isomorphism as a composition of isomorphisms. $\square$

22. THE PROOF OF EXCISION

In this section we prove Theorem 20.2. A main tool used in the proof is the *barycentric subdivision of simplices*:

Since $\Delta^0 = \{1\}$, it can not be divided any further.

The one-dimensional standard simplex is $[e_0, e_1]$, where $e_0 = (1, 0)$ and $e_1 = (0, 1)$. The *barycenter* b of Δ^1 is the midpoint of the interval $[e_0, e_1]$. The barycentric subdivision of Δ^1 consists of the simplices $[e_0, b]$ and $[b, e_1]$.

The two-dimensional standard simplex is $\Delta^2 = [e_0, e_1, e_2]$. Let

- (1) b_0 be the barycenter of $[e_1, e_2]$,
- (2) b_1 be the barycenter of $[e_0, e_2]$,
- (3) b_2 be the barycenter of $[e_0, e_1]$,
- (4) b be the barycenter of $[e_0, e_1, e_2]$.

Notice that e_0 , e_1 and e_2 are barycenters of 0-faces (themselves), b_1 , b_2 and b_3 are barycenters of 1-faces, and b is the barycenter of Δ^2 . Thus each vertex of the barycentric subdivision of Δ^2 can be considered as the barycenter b^σ of a face σ of Δ^2 .

For faces τ and σ of Δ^2 , write $\tau < \sigma$, if τ is a proper face of σ . Then $\{b^\tau, b^\sigma, b^\delta\}$ forms a triangle exactly when $\tau < \sigma < \delta$ or $\tau < \sigma < \delta$ etc., and it follows that there are exactly $3 \cdot 2 = 3!$ triangles in the barycentric subdivision of Δ^2 .

Definition 22.1. Let Σ^n be an affine n -simplex. The *barycentric subdivision* $\text{Sd}\Sigma^n$ is a family of affine n -simplices defined inductively for $n \geq 0$:

- (1) $\text{Sd}\Sigma^0 = \Sigma^0$.
- (2) Let $\varphi_0, \varphi_1, \dots, \varphi_{n+1}$ be the n -faces of Σ^{n+1} and let b be the barycenter of Σ^{n+1} . Then $\text{Sd}\Sigma^{n+1}$ consists of all the $(n+1)$ -simplices spanned by b and n -simplices Sd_{φ_i} , $i = 0, \dots, n+1$.

Then Σ^n is the union of the n -simplices in $\text{Sd}\Sigma^n$.

Exercise 22.2. Show that $\text{Sd}\Sigma^n$ consists of exactly $(n+1)!$ n -simplices.

Definition 22.3. Let E be a convex subset of a euclidean space. The barycentric subdivision of E is a homomorphism

$$\text{Sd}_n: S_n(E) \rightarrow S_n(E)$$

defined inductively on generators $\tau: \Delta^n \rightarrow E$ as follows:

- (1) If $n = 0$, then $\text{Sd}_0(\tau) = \tau$.
- (2) If $n > 0$, then

$$\text{Sd}_n(\tau) = \tau(b_n) \cdot \text{Sd}_{n-1}(\partial\tau),$$

where b_n is the barycenter of Δ^n and

$$\begin{aligned} \tau(b_n) \cdot \text{Sd}_{n-1}(\partial\tau) &= \tau(b_n) \cdot \text{Sd}_{n-1}\left(\sum_{j=0}^n (-1)^j \tau \circ e^j\right) \\ &\stackrel{\text{def}}{=} \sum_{j=0}^n (-1)^j \tau(b_n) \cdot \text{Sd}_{n-1}(\tau \circ e^j). \end{aligned}$$

Notice that this type of maps were already used to calculate the homology groups of convex subsets of $\mathbb{R}^n$, see Example 17.5.

Recall that for a convex subset X of $\mathbb{R}^n$, a point $x_0 \in X$ and a continuous map $T: \Delta^n \rightarrow X$ we defined

$$x_0 \cdot T: \Delta^{n+1} \rightarrow X$$

by setting

$$(x_0 \cdot T)(t_0, \dots, t_{n+1}) = \begin{cases} x_0, & \text{if } t_0 = 1, \\ t_0(x_0) + (1 - t_0)T\left(\frac{t_1}{1-t_0}, \dots, \frac{t_{n+1}}{1-t_0}\right), & \text{if } 0 \leq t_0 < 1. \end{cases}$$

Thus, for $\tau: \Delta^n \rightarrow E$,

$$\begin{aligned} &\tau(b_n) \cdot \text{Sd}_{n-1}(\tau \circ e^j)(t_0, \dots, t_n) \\ &= \begin{cases} \tau(b_n), & \text{if } t_0 = 1, \\ t_0 \tau(b_n) + (1 - t_0) \text{Sd}_{n-1}(\tau \circ e^j)\left(\frac{t_1}{1-t_0}, \dots, \frac{t_{n+1}}{1-t_0}\right), & \text{if } 0 \leq t_0 < 1. \end{cases} \end{aligned}$$

Example 22.4. Let $n = 1$, and let $\tau = \delta^1 = \text{id}: \Delta^1 \rightarrow \Delta^1$. Then $b_n = (\frac{1}{2}, \frac{1}{2})$. Also,

$$\partial\delta^1 = \delta^1 \circ e^0 - \delta^1 \circ e^1,$$

where

$$\delta^1 \circ e^0(1) = \delta^1(0, 1) = (0, 1) \text{ and } \delta^1 \circ e^1(1) = \delta^1(1, 0) = (1, 0).$$

Here

$$\delta^1 \circ e^0, \delta^1 \circ e^1: \Delta^0 \rightarrow E,$$

so that

$$\text{Sd} \circ (\delta^1 \circ e^0) = \delta^1 \circ e^0 \text{ and } \text{Sd} \circ (\delta^1 \circ e^1) = \delta^1 \circ e^1.$$

Then

$$\begin{aligned} & \delta^1(b_n) \cdot \text{Sd}_0(\delta^1 \circ e^0)(t_0, t_1) \\ &= ((\frac{1}{2}, \frac{1}{2}) \cdot (\delta^1 \circ e^0))(t_0, t_1) \\ &= \begin{cases} (\frac{1}{2}, \frac{1}{2}), & \text{if } t_0 = 1, \\ t_0(\frac{1}{2}, \frac{1}{2}) + (1 - t_0)(\delta^1 \circ e^0)(\underbrace{\frac{t_1}{1 - t_0}}_{=1}), & \text{if } 0 \leq t_0 < 1 \end{cases} \\ &= \begin{cases} (\frac{1}{2}, \frac{1}{2}), & \text{if } t_0 = 1, \\ t_0(\frac{1}{2}, \frac{1}{2}) + (1 - t_0)\delta^1(0, 1) = t_0(\frac{1}{2}, \frac{1}{2}) + (1 - t_0)(0, 1), & \text{if } 0 \leq t_0 < 1. \end{cases} \end{aligned}$$

Similarly,

$$\begin{aligned} & \delta^1(b_n) \cdot \text{Sd}_0(\delta^1 \circ e^1)(t_0, t_1) \\ &= \begin{cases} (\frac{1}{2}, \frac{1}{2}), & \text{if } t_0 = 1, \\ t_0(\frac{1}{2}, \frac{1}{2}) + (1 - t_0)(\delta^1 \circ e^1)(1) = t_0(\frac{1}{2}, \frac{1}{2}) + (1 - t_0)(1, 0), & \text{if } 0 \leq t_0 < 1. \end{cases} \end{aligned}$$

Let next X be any topological space. The n^{th} barycentric subdivision of X , $n \geq 0$, is the homomorphism

$$\text{Sd}_n: S_n(X) \rightarrow S_n(X),$$

defined on the generators $\sigma: \Delta^n \rightarrow X$ of $S_n(X)$ by

$$\text{Sd}_n(\sigma) = \sigma_{\#}\text{Sd}_n(\delta^n),$$

where $\delta^n: \Delta^n \rightarrow \Delta^n$ is the identity map:

$$S_n(\Delta^n) \xrightarrow{\text{Sd}_n} S_n(\Delta^n) \xrightarrow{\sigma_{\#}} S_n(X), \quad \delta^n \mapsto \text{Sd}_n(\delta^n) \mapsto \sigma_{\#}\text{Sd}_n(\delta^n).$$

Example 22.5. By Example 22.4, $\text{Sd}_1(\delta^1) = T_0 - T_1$, where

$$T_0: \Delta^1 \rightarrow \Delta^1, (t_0, t_1) \mapsto t_0(\frac{1}{2}, \frac{1}{2}) + (1 - t_0)(0, 1),$$

and

$$T_1: \Delta^1 \rightarrow \Delta^1, (t_0, t_1) \mapsto t_0(\frac{1}{2}, \frac{1}{2}) + (1 - t_0)(1, 0).$$

Let $\sigma: \Delta^1 \rightarrow X$. Then

$$\text{Sd}_1(\sigma) = \sigma_{\#}\text{Sd}_1(\delta^1) = \sigma_{\#}(T_0 - T_1) = \sigma \circ T_0 - \sigma \circ T_1.$$

Lemma 22.6. *Let $f: X \rightarrow Y$ be continuous. Then the diagram*

$$\begin{array}{ccc} S_n(X) & \xrightarrow{\text{Sd}_n} & S_n(X) \\ f_{\#} \downarrow & & \downarrow f_{\#} \\ S_n(Y) & \xrightarrow{\text{Sd}_n} & S_n(Y) \end{array}$$

commutes, for every $n \geq 0$.

Proof. Let $\sigma: \Delta^n \rightarrow X$ be a generator of $S_n(X)$. Then

$$\begin{aligned} (f_{\#} \circ \text{Sd}_n)(\sigma) &= f_{\#}(\sigma_{\#} \text{Sd}_n(\delta^n)) \\ &= (f_{\#} \circ \sigma_{\#})(\text{Sd}_n(\delta^n)) \\ &= (f \circ \sigma)_{\#}(\text{Sd}_n(\delta^n)) \\ &= \text{Sd}_n(f \circ \sigma) \\ &= (\text{Sd} \circ f_{\#})(\sigma). \end{aligned}$$

□

Lemma 22.7. *The map $\text{Sd}: S_*(X) \rightarrow S_*(X)$ is a chain map.*

Proof. First, let's assume X is convex. Let $\tau: \Delta^n \rightarrow X$ be a singular n -simplex. We show that $\text{Sd}_{n-1}\partial_n\tau = \partial_n\text{Sd}_n\tau$. The proof is done by induction on n .

Let $n = 0$. Then $S_{-1}(X) = 0$, thus $\partial_0 = 0$ and $\text{Sd}_{-1} = 0$. Hence $\text{Sd}_{-1} \circ \partial_0 = \partial_0 \circ \text{Sd}_0$.

Let then $n > 0$.

$$\begin{aligned} \partial_n \text{Sd}_n \tau &= \partial_n(\tau_n(b_n) \cdot \text{Sd}_{n-1}(\partial_n \tau)) && \text{(definition of } \text{Sd}_n) \\ &= \text{Sd}_{n-1} \partial_n \tau - \tau(b_n) \cdot ((\partial_{n-1} \text{Sd}_{n-1}) \partial_n \tau) && \text{(see p. 63)} \\ &= \text{Sd}_{n-1} \partial_n \tau - \tau(b_n) \cdot ((\text{Sd}_{n-2} \underbrace{\partial_{n-1}}_0) \partial_n \tau) && \text{(induction)} \\ &= \text{Sd}_{n-1} \partial_n \tau. \end{aligned}$$

This proves the claim for convex X .

Let then X be any space, not necessarily convex. Let $\sigma: \Delta^n \rightarrow X$ be a singular simplex. Then

$$\begin{aligned} \partial_n \text{Sd}_n(\sigma) &= \partial_n \sigma_{\#} \text{Sd}_n(\delta^n) && \text{(definition of } \text{Sd}_n) \\ &= \sigma_{\#} \partial_n \text{Sd}_n(\delta^n) && (\sigma_{\#} \text{ is a chain map)} \\ &= \sigma_{\#} \text{Sd}_{n-1} \partial_n(\delta^n) && (\Delta^n \text{ is convex)} \\ &= \text{Sd}_{n-1} \sigma_{\#} \partial_n(\delta^n) && \text{(Lemma 22.6)} \\ &= \text{Sd}_{n-1} \partial_n \sigma_{\#}(\delta^n) && (\sigma_{\#} \text{ is a chain map)} \\ &= \text{Sd}_{n-1} \partial_n(\sigma). && (\sigma_{\#}(\delta^n) = \sigma) \end{aligned}$$

□

Lemma 22.8. *For each $n \geq 0$, $H_n(\text{Sd}): H_n(X) \rightarrow H_n(X)$ is the identity.*

Proof. By Proposition 16.3, it suffices to prove that Sd is chain homotopic to $id: S_*(X) \rightarrow S_*(X)$. Therefore, we will construct homomorphisms $T_n: S_n(X) \rightarrow S_{n+1}(X)$ satisfying

$$\partial_{n+1}T_n + T_{n-1}\partial_n = id_n - Sd_n.$$

First, let's assume that X is convex, and let's construct the T_n by induction on n . For $n = 0$, let

$$T_0 = 0: S_0(X) \rightarrow S_1(X).$$

Then, for a 0-simplex σ ,

$$\partial_1 T_0 \sigma = 0 \text{ and } id(\sigma) - Sd_0(\sigma) = \sigma - \sigma = 0.$$

Let then $n > 0$. Then T_n should satisfy

$$\partial_{n+1}T_n\gamma + T_{n-1}\partial_n\gamma = \gamma - Sd_n(\gamma),$$

for any $\gamma \in S_n(X)$. Equivalently, should be

$$\partial_{n+1}T_n\gamma = \gamma - Sd_n(\gamma) - T_{n-1}\partial_n\gamma. \quad (\clubsuit)$$

The right side of $(\clubsuit)$ is

$$\begin{aligned} & \partial_n(\gamma - Sd_n(\gamma) - T_{n-1}\partial_n\gamma) \\ &= \partial_n\gamma - \partial_n Sd_n(\gamma) - \partial_n T_{n-1}\partial_n\gamma \\ &= \partial_n\gamma - \partial_n Sd_n(\gamma) - (id_{n-1} - Sd_{n-1} - T_{n-2}\partial_{n-1})\partial_n\gamma \quad (\text{by induction}) \\ &= \partial_n\gamma - \partial_n Sd_n(\gamma) - \partial_n\gamma + Sd_{n-1}\partial_n\gamma + T_{n-2}\underbrace{\partial_{n-1}\partial_n\gamma}_0 \\ &= \underbrace{(Sd_{n-1}\partial_n - \partial_n Sd_n)}_{0, \text{ since } Sd \text{ is a chain map}}(\gamma) = 0. \end{aligned}$$

Thus $\gamma - Sd_n(\gamma) - T_{n-1}\partial_n\gamma \in Z_n(X)$. In the end of Example 17.5 we showed that

$$\partial_{n+1}(b \cdot \gamma') = \gamma',$$

for $\gamma' \in Z_n(X)$. Thus, let's define T_n by

$$T_n(\gamma) = b \cdot (\gamma - Sd_n(\gamma) - T_{n-1}\partial_n\gamma),$$

where $b \in X$. It follows that

$$\begin{aligned} \partial_{n+1}T_n(\gamma) &= \gamma - Sd(\gamma) - T_{n-1}\partial_n\gamma \\ &= \text{the right side of } (\clubsuit). \end{aligned}$$

Thus we have constructed the T_n for convex spaces X .

Let then X be any topological space, not necessarily convex. Let $\sigma: \Delta^n \rightarrow X$ be an n -simplex. Define

$$T_n(\sigma) = \sigma_{\#} \left(\underbrace{T_n(\delta^n)}_{\in S_{n+1}(\Delta^n)} \right) \in S_{n+1}(X),$$

and extend T_n by linearity,

$$T_n\left(\sum_i n_i \sigma_i\right) = \sum_i T_n(\sigma_i),$$

for $\sigma_i: \Delta^n \rightarrow X$. Then

$$\begin{aligned}
& \partial_{n+1}T_n\sigma + T_{n-1}\partial_n\sigma \\
& \stackrel{\text{def.}}{=} \partial_{n+1}\sigma_{\#}T_n(\delta^n) + T_{n-1}\partial_n\sigma_{\#}(\delta^n) \\
& = \sigma_{\#}\partial_{n+1}T_n(\delta^n) + T_{n-1}\partial_n\sigma_{\#}(\delta^n) \quad (\sigma_{\#} \text{ is a chain map}) \\
& = \sigma_{\#}(\delta^n - \text{Sd}_n(\delta^n) - T_{n-1}\partial_n(\delta^n)) + T_{n-1}\partial_n\sigma_{\#}(\delta^n) \\
& = \sigma - \underbrace{\sigma_{\#}\text{Sd}_n(\delta^n)}_{=\text{Sd}_n(\sigma)} - \sigma_{\#}T_{n-1}\partial_n(\delta^n) + T_{n-1}\sigma_{\#}\partial_n(\delta^n) \quad (\text{the claim holds for } \Delta^n) \\
& = \sigma - \text{Sd}_n(\sigma),
\end{aligned}$$

since

$$\sigma_{\#}T_{n-1} = T_{n-1}\sigma_{\#}$$

by the following commutative diagram

$$\begin{array}{ccc}
S_n(\Delta^n) & \xrightarrow{\sigma_{\#}} & S_n(X) \\
T_n \downarrow & & \downarrow T_n \\
S_{n+1}(\Delta^n) & \xrightarrow[\sigma_{\#}]{} & S_{n+1}(X),
\end{array}$$

where $\sigma: \Delta^n \rightarrow X$, $c: \Delta^n \rightarrow \Delta^n$,

$$T_n\sigma_{\#}(c) = T_n(\sigma \circ c) = (\sigma \circ c)_{\#}T_n(\delta^n)$$

and also

$$\sigma_{\#}T_n(c) = \sigma_{\#} \circ c_{\#}T_n(\delta^n) = (\sigma \circ c)_{\#}T_n(\delta^n).$$

□

Corollary 22.9. *Let $q \geq 0$ and let $z \in Z_n(X)$. Then $[z] = [\text{Sd}^q z]$.*

Proof. Since $H_n(\text{Sd}): H_n(X) \rightarrow H_n(X)$ is the identity, it follows that $[\text{Sd}z] = [z]$. Thus $[\text{Sd}^2 z] = [\text{Sd}z] = [z]$ and, inductively $[\text{Sd}^q z] = [z]$. □

Recall that the diameter of an n -simplex $S = [p_0, \dots, p_n]$ is

$$\begin{aligned}
\text{diam} & \stackrel{\text{def.}}{=} \sup\{\|u - v\| \mid u, v \in S\} \\
& \stackrel{5.21}{=} \sup_{i,j}\{\|p_i - p_j\|\}.
\end{aligned}$$

For $\delta^n = \text{id}: \Delta^n \rightarrow \Delta^n$, define

$$\text{diam}(\delta^n) = \text{diam}(\delta^n(\Delta^n)) = \text{diam}(\Delta^n).$$

If $\text{Sd}(\delta^n) = \sum m_i \sigma_i$, define

$$\text{diam}(\sigma_i) = \text{diam}(\sigma_i(\Delta^n)).$$

Lemma 22.10. *Let $\delta^n = \text{id}: \Delta^n \rightarrow \Delta^n$. Let $\text{Sd}(\delta^n) = \sum m_i \sigma_i$. Then*

$$\text{diam}(\sigma_i) \leq \frac{n}{n+1} \text{diam}(\delta^n),$$

for every i .

Proof. The image $\sigma_i(\Delta^n)$ is an n -simplex for every i , the vertices of $\sigma_i(\Delta^n)$ are barycenters of the faces of the original simplex, for example it can be that the vertices are

$$e_0, b_{01} = \frac{1}{2}(e_0 + e_1), b_{012} = \frac{1}{3}(e_0 + e_1 + e_2), \dots, b_{012\dots n} = \frac{1}{n+1}(e_0 + \dots + e_n).$$

Notice that not any arbitrary choice of barycenters gives a simplex in the barycentric subdivision, they must be compatible. For example: $[e_0]$ is a proper face of $[e_0, b_{01}]$, $[e_0, b_{01}]$ is a proper face of $[e_0, b_{01}, b_{012}]$ etc. (Thus the barycentric subdivision of Δ^n has $(n+1)!$ small simplices.)

Notice also that after subdividing the standard n -simplex Δ^n we get smaller simplices that will be subdivided again. Therefore, in the following calculation the vertices do not have to be the vertices of the standard n -simplex. Let's denote the vertices of Δ^n by $w_1, \dots, w_{n+1}$. Then each vertex of $\sigma_i(\Delta^n)$ is of the form

$$\frac{1}{k} \sum_{j=1}^k w_j,$$

where $1 \leq j \leq n+1$. Then, showing that

$$\left\| \frac{1}{k} \sum_{j=1}^k w_j - \frac{1}{m} \sum_{j=1}^m w_j \right\| \leq \frac{n}{n+1} \underbrace{\max_{l,j} \{\|w_l - w_j\|\}}_{\text{diam}(\Delta^n)},$$

for all $1 \leq m < k \leq n+1$, will imply that $\text{diam}(\sigma_i) \leq \frac{n}{n+1} \text{diam}(\delta^n)$. Therefore, let's make an estimation:

$$\begin{aligned} & \left\| \frac{1}{m} \sum_{j=1}^m w_j - \frac{1}{k} \sum_{j=1}^k w_j \right\| \\ &= \left\| \frac{1}{m} \sum_{j=1}^m w_j - \frac{1}{k} \sum_{j=1}^m w_j - \frac{1}{k} \sum_{j=m+1}^k w_j \right\| \\ &= \left\| \frac{k-m}{km} \sum_{j=1}^m w_j - \frac{1}{k} \sum_{j=m+1}^k w_j \right\| \\ &= \frac{k-m}{k} \left\| \frac{1}{m} \sum_{j=1}^m w_j - \frac{1}{k-m} \sum_{j=m+1}^k w_j \right\|, \end{aligned}$$

where each sum in the bottom term is in the convex hull of the w_j , and therefore in Δ^n . Thus it follows that

$$\left\| \frac{1}{m} \sum_{j=1}^m w_j - \frac{1}{k} \sum_{j=1}^k w_j \right\| \leq \frac{k-m}{m} \text{diam}(\delta^n).$$

The function $f: [0, \infty) \rightarrow [0, \infty)$, $x \mapsto \frac{x}{x+1}$, is increasing. Thus

$$\frac{k-m}{k} \leq \frac{k-1}{k} = f(k-1) \leq f(n) = \frac{n}{n+1},$$

which implies that

$$\left\| \frac{1}{m} \sum_{j=1}^m w_j - \frac{1}{k} \sum_{j=1}^k w_j \right\| \leq \frac{n}{n+1} \text{diam}(\delta^n),$$

and furthermore that

$$\text{diam}(\sigma_i) \leq \frac{n}{n+1} \text{diam}(\delta^n).$$

□

Corollary 22.11. *Let $\delta^n = \text{id}: \Delta^n \rightarrow \Delta^n$, and let $\text{Sd}^q(\delta^n) = \sum a_i \sigma_i$. Then*

$$\text{diam}(\sigma_i) \leq \left(\frac{n}{n+1} \right)^q \text{diam}(\delta^n),$$

for all i .

Proof. Apply the proof of Lemma 22.10 repeatedly. □

Lemma 22.12. *Let X be a topological space and let X_1 and X_2 be subspaces of X with $X = \mathring{X}_1 \cup \mathring{X}_2$. Let $\gamma: \Delta^n \rightarrow X$ be a continuous map. Then there is $p \geq 1$, with*

$$\text{Sd}^p(\gamma) \in S_n(X_1) + S_n(X_2).$$

Proof. Since $\gamma: \Delta^n \rightarrow X$ is continuous, it follows that the sets $\gamma^{-1}(\mathring{X}_1)$ and $\gamma^{-1}(\mathring{X}_2)$ form an open cover for Δ^n . Now, Δ^n is a compact metric space. By the Lebesgue number theorem (Theorem 10.1), there exists $\lambda > 0$ such that when ever $x, y \in \Delta^n$, $\|x - y\| < \lambda$, then

$$\{x, y\} \subset \gamma^{-1}(\mathring{X}_1) \text{ or } \{x, y\} \subset \gamma^{-1}(\mathring{X}_2).$$

Let $p \geq 1$ be such that

$$\left(\frac{n}{n+1} \right)^p \text{diam} \Delta^n < \lambda.$$

Write $\text{Sd}^p(\delta^n) = \sum m_j \sigma_j$. By Corollary 22.11,

$$\text{diam}(\sigma_j) \leq \left(\frac{n}{n+1} \right)^p \underbrace{\text{diam}(\delta^n)}_{=\text{diam}(\Delta^n)} < \lambda,$$

for any singular simplex σ_j in the linear combination of $\text{Sd}^p(\delta^n)$. Then $\text{diam}(\sigma_j(\Delta^n)) < \lambda$, for every j . Thus, for every j ,

$$\sigma_j(\Delta^n) \subset \gamma^{-1}(\mathring{X}_1) \text{ or } \sigma_j(\Delta^n) \subset \gamma^{-1}(\mathring{X}_2).$$

Now,

$$\text{Sd}^p \gamma = \gamma_{\#} \text{Sd}^p(\delta^n) = \gamma_{\#} \left(\sum m_j \sigma_j \right) = \sum m_j \gamma_{\#} \sigma_j.$$

Thus

$$\gamma_{\#} \sigma_j(\Delta^n) \subset \mathring{X}_1 \subset X_1 \text{ or } \gamma_{\#} \sigma_j(\Delta^n) \subset \mathring{X}_2 \subset X_2,$$

for every j . Thus, by collecting terms, we can write $\text{Sd}^p \gamma = \gamma_1 + \gamma_2$, where $\gamma_1 \in S_n(X_1)$ and $\gamma_2 \in S_n(X_2)$. □

Lemma 22.13. *Let X be a topological space and let X_1 and X_2 be subspaces of X . Assume the inclusion*

$$S_*(X_1) + S_*(X_2) \rightarrow S_*(X)$$

induces isomorphisms in homology. Then excision holds for the subspaces X_1 and X_2 of X , i.e., the inclusion

$$j: (X_1, X_1 \cap X_2) \hookrightarrow (X_1 \cup X_2, X_2) = (X, X_2)$$

induces isomorphisms

$$j_*: H_n(X_1, X_1 \cap X_2) \rightarrow H_n(X, X_2),$$

for all $n \geq 0$.

Proof. There is a short exact sequence of chain complexes:

$$0 \longrightarrow \underbrace{S_*(X_1) + S_*(X_2)}_K \longrightarrow \underbrace{S_*(X)}_L \longrightarrow \underbrace{S_*(X)/(S_*(X_1) + S_*(X_2))}_M \longrightarrow 0,$$

which induces a long exact sequence in homology:

$$\cdots \longrightarrow H_n(K) \xrightarrow{f_n} H_n(L) \longrightarrow H_n(M) \longrightarrow H_{n-1}(K) \xrightarrow{f_{n-1}} H_{n-1}(L) \longrightarrow \cdots$$

By assumption the sequence has isomorphisms f_n , for all n . It follows that

$$H_*(S_*(X)/(S_*(X_1) + S_*(X_2))) = H_*(M) = 0,$$

for all n . There also is a short exact sequence of chain complexes:

$$0 \longrightarrow \underbrace{\frac{S_*(X_1) + S_*(X_2)}{S_*(X_2)}}_{K'} \xrightarrow{k} \underbrace{\frac{S_*(X)}{S_*(X_2)}}_{L'} \longrightarrow \underbrace{S_*(X)/(S_*(X_1) + S_*(X_2))}_M \longrightarrow 0,$$

which induces a long exact sequence in homology:

$$\cdots \longrightarrow H_n(K') \xrightarrow{H_n(k)} H_n(L') \longrightarrow \underbrace{H_n(M)}_0 \longrightarrow H_{n-1}(K') \longrightarrow H_{n-1}(L') \longrightarrow \cdots$$

Since $H_n(M) = 0$, for every n , it follows that

$$k_*: H_*(K') \rightarrow H_*(L')$$

is an isomorphism for all $*$. Consider the diagram

$$\begin{array}{ccc} S_*(X_1)/S_*(X_1 \cap X_2) & & \\ \downarrow l & \searrow j & \\ (S_*(X_1) + S_*(X_2))/S_*(X_2) & \xrightarrow[k]{} & S_*(X)/S_*(X_2) \end{array}$$

where j is induced by the inclusion

$$(X_1, X_1 \cap X_2) \hookrightarrow (X, X_2).$$

The diagram commutes and l is an isomorphism of chain complexes. Since $j = kl$, it follows that

$$j_* = k_* l_*: \underbrace{H_*(S_*(X_1)/S_*(X_1 \cap X_2))}_{H_*(X_1, X_1 \cap X_2)} \rightarrow \underbrace{H_*(S_*(X)/S_*(X_2))}_{H_*(X, X_2)}.$$

Since k_* and l_* are isomorphisms, it follows that j_* is an isomorphism. Therefore, excision holds for X_1 and X_2 in X . $\square$

We are ready to prove that singular homology satisfies excision. Since the two versions for excision are equivalent, it suffices to prove one of them. Recall the second version (Theorem 20.2):

Theorem 20.2 (*Excision 2.*) *Let X be a topological space and let X_1 and X_2 be subspaces of X with $X = \overset{\circ}{X}_1 \cup \overset{\circ}{X}_2$. Then the inclusion*

$$j: (X_1, X_1 \cap X_2) \hookrightarrow (X_1 \cup X_2, X_2) = (X, X_2)$$

induces isomorphisms

$$j_*: H_n(X_1, X_1 \cap X_2) \rightarrow H_n(X, X_2),$$

for every n .

Proof. By lemma 22.13, it suffices to show that the inclusion

$$S_*(X_1) + S_*(X_2) \rightarrow S_*(X)$$

induces isomorphisms

$$\theta_n: H_n(S_*(X_1) + S_*(X_2)) \rightarrow H_n(S_*(X)) = H_n(X),$$

for all n . Let $\gamma_1 \in S_n(X_1)$, $\gamma_2 \in S_n(X_2)$, and assume $\gamma_1 + \gamma_2$ is a cycle in $S_n(X)$. Then $\gamma_1 + \gamma_2$ is also a cycle in $S_n(X_1) + S_n(X_2)$. Denote the homology class of $\gamma_1 + \gamma_2$ in $S_n(X_1) + S_n(X_2)$ by $[\gamma_1 + \gamma_2]_{\text{sub}}$ and in $S_n(X)$ by $[\gamma_1 + \gamma_2]$, as usual. Then

$$\theta_n: [\gamma_1 + \gamma_2]_{\text{sub}} \mapsto [\gamma_1 + \gamma_2].$$

We first show that θ_n is surjective: Let $z = \sum m_i z_i \in Z_n(X) \subset S_n(X)$, where $z_i: \Delta^n \rightarrow X$ are continuous for all (finitely many) i . Then, by Lemma 22.12, there is $q \geq 1$ such that

$$\text{Sd}^q(z_i) = \gamma_i^1 + \gamma_i^2,$$

where $\gamma_i^1 \in S_n(X_1)$ and $\gamma_i^2 \in S_n(X_2)$, for every i . Since z is an n -cycle and since Sd^q is a chain map, it follows that

$$\text{Sd}^q z = \text{Sd}^q \left(\sum m_i z_i \right) = \sum m_i (\gamma_i^1 + \gamma_i^2)$$

is an n -cycle, both in $S_*(X)$ and in $S_*(X_1) + S_*(X_2)$. Therefore,

$$[\sum m_i (\gamma_i^1 + \gamma_i^2)]_{\text{sub}} \in H_n(S_*(X_1) + S_*(X_2))$$

and

$$\begin{aligned} \theta_n([\sum m_i (\gamma_i^1 + \gamma_i^2)]_{\text{sub}}) &= [\sum m_i (\gamma_i^1 + \gamma_i^2)] \\ &= [\text{Sd}^q(z)] = [z]. \end{aligned}$$

It follows that θ_n is surjective.

It remains to show that θ_n is injective: Assume

$$\theta_n[\gamma_1 + \gamma_2]_{\text{sub}} = 0 = [\gamma_1 + \gamma_2].$$

Then there exists $\beta \in S_{n+1}(X)$ with $\partial\beta = \gamma_1 + \gamma_2$. It follows from Lemma 22.12 that there is $q \geq 1$ satisfying $\text{Sd}^q\beta = \beta_1 + \beta_2$, where $\beta_1 \in S_{n+1}(X_1)$ and $\beta_2 \in S_{n+1}(X_2)$. Then

$$\partial(\beta_1 + \beta_2) = \partial\text{Sd}^q\beta = \text{Sd}^q\partial\beta = \text{Sd}^q(\gamma_1 + \gamma_2),$$

where the second equality holds since Sd^q is a chain map. Thus $[\text{Sd}^q(\gamma_1 + \gamma_2)]_{\text{sub}} = 0$. Let $j_i: X_i \hookrightarrow X$, $i = 1, 2$, be the inclusions. By Lemma 22.6, the diagram

$$\begin{array}{ccc} S_n(X_i) & \xrightarrow{\text{Sd}_n} & S_n(X_i) \\ (j_i)_\# \downarrow & & \downarrow (j_i)_\# \\ S_n(X) & \xrightarrow{\text{Sd}_n} & S_n(X) \end{array}$$

commutes for $i = 1, 2$. Thus $\text{Sd}: S_*(X) \rightarrow S_*(X)$ takes $S_*(X_1)$ into $S_*(X_1)$ and $S_*(X_2)$ into $S_*(X_2)$. Therefore, Sd takes the subcomplex $S_*(X_1) + S_*(X_2)$ into $S_*(X_1) + S_*(X_2)$. Similarly, the diagram

$$\begin{array}{ccc} S_n(X_i) & \xrightarrow{T_n} & S_{n+1}(X_i) \\ (j_i)_\# \downarrow & & \downarrow (j_i)_\# \\ S_n(X) & \xrightarrow{T_n} & S_{n+1}(X) \end{array}$$

commutes for $i = 1, 2$. Thus the chain homotopy $\{T_n: S_n(X) \rightarrow S_{n+1}(X)\}$ from id to Sd restricts to chain homotopies $\{T_n^1: S_n(X_1) \rightarrow S_{n+1}(X)\}$ and $\{T_n^2: S_n(X_2) \rightarrow S_{n+1}(X_2)\}$. Hence

$$\gamma_1 - \text{Sd}^q\gamma_1 = (T^1\partial + \partial T^1)\gamma_1$$

and

$$\gamma_2 - \text{Sd}^q\gamma_2 = (T^2\partial + \partial T^2)\gamma_2.$$

Thus

$$\gamma_1 + \gamma_2 - \text{Sd}^q(\gamma_1 + \gamma_2) = \underbrace{T^1\partial\gamma_1 + T^2\partial\gamma_2}_{T\partial(\gamma_1 + \gamma_2)} + \partial(T^1\gamma_1 + T^2\gamma_2).$$

Now,

$$\partial(T^1\gamma_1 + T^2\gamma_2) \in B_n(S_*(X_1) + S_*(X_2)).$$

Since $[\gamma_1 + \gamma_2] = 0$, it follows that $T\partial(\gamma_1 + \gamma_2) = 0$. Thus

$$\begin{aligned} [\gamma_1 + \gamma_2]_{\text{sub}} &= [\text{Sd}^q(\gamma_1 + \gamma_2) + \partial(T^1\gamma_1 + T^2\gamma_2)]_{\text{sub}} \\ &= [\text{Sd}^q(\gamma_1 + \gamma_2)]_{\text{sub}} \\ &= 0. \end{aligned}$$

It follows that θ_n is injective, which completes the proof. $\square$

23. HOMOLOGY OF A WEDGE SUM

Let X_α , $\alpha \in I$, be topological spaces. Let $x_\alpha \in X_\alpha$ be a basepoint of X_α , for all α . The *wedge sum* (or just the *wedge*) of the X_α is

$$\bigvee_{\alpha} X_{\alpha} = \bigsqcup_{\alpha} X_{\alpha} / \sim,$$

where $\bigsqcup$ denotes a *disjoint union*, and $\sim$ is the equivalence relation on $\bigsqcup_{\alpha} X_{\alpha}$ obtained by identifying all the basepoints x_α with each others. Write

$$X = \bigsqcup_{\alpha} X_{\alpha} \text{ and } A = \bigsqcup_{\alpha} \{x_{\alpha}\}.$$

Then

$$X/A = \bigvee_{\alpha} X_{\alpha}.$$

Proposition 23.1. *Let X_α , $\alpha \in I$, be path-connected topological spaces with basepoints x_α , respectively. Assume each x_α is a strong deformation retract of an open subset of X_α . Then, for all n ,*

$$\tilde{H}_n\left(\bigvee_{\alpha} X_{\alpha}\right) \cong \bigoplus_{\alpha} \tilde{H}_n(X_{\alpha}).$$

Proof. For every $\alpha \in I$, let $F_\alpha: U_\alpha \times I \rightarrow U_\alpha$ be a strong deformation retraction to x_α . Thus

- (1) $F_\alpha(x, 0) = x$, for all $x \in U_\alpha$.
- (2) $F_\alpha(x, 1) = x_\alpha$, for all $x \in U_\alpha$.
- (3) $F_\alpha(x_\alpha, t) = x_\alpha$, for all $t \in I$.

Let

$$F: \bigsqcup_{\alpha} U_{\alpha} \times I \rightarrow \bigsqcup_{\alpha} U_{\alpha}, \quad (x, t) \mapsto F_{\alpha}(x, t) \quad \forall (x, t) \in U_{\alpha} \times I.$$

Then

- (1) $F(x, 0) = x$, for all $x \in \bigsqcup_{\alpha} U_{\alpha}$,
- (2) $F(x, 1) \in A = \bigsqcup \{x_{\alpha}\}$, for all $x \in \bigsqcup_{\alpha} U_{\alpha}$,
- (3) $F(x_{\alpha}, t) = x_{\alpha}$, for all x_{α} and for all t .

Thus F strongly deformation retracts $\bigsqcup_{\alpha} U_{\alpha}$ to A . It follows that, for all $n \geq 0$,

$$\begin{aligned} \tilde{H}_n\left(\bigvee_{\alpha} X_{\alpha}\right) &= \tilde{H}_n\left(\bigsqcup_{\alpha} X_{\alpha}/A\right) \\ &\cong H_n\left(\bigsqcup_{\alpha} X_{\alpha}, A\right) && \text{(Theorem 21.7)} \\ &\cong \bigoplus_{\alpha} H_n(X_{\alpha}, A \cap X_{\alpha}) && \text{(Theorem 19.5)} \\ &= \bigoplus_{\alpha} H_n(X_{\alpha}, x_{\alpha}) \\ &\cong \bigoplus_{\alpha} \tilde{H}_n(X_{\alpha}). && \text{(Theorem 19.9)} \end{aligned}$$

□

24. JORDAN SEPARATION THEOREM AND INVARIANCE OF DOMAIN

Definition 24.1. Let X be a topological space. Assume one-point sets are closed in X . The space X is said to be *normal*, if for each pair A, B of disjoint closed subsets of X , there exist disjoint open subsets of X containing A and B , respectively.

It follows immediately from Definition 24.1, that every normal space is Hausdorff.

Theorem 24.2. (*Tietze extension theorem*) Let X be a normal space, and let A be a closed subspace of X . Then:

- (1) Any continuous map $A \rightarrow [a, b]$ ($a, b \in \mathbb{R}$, $a < b$) may be extended to a continuous map $X \rightarrow [a, b]$.
- (2) Any continuous map $A \rightarrow \mathbb{R}$ may be extended to a continuous map $X \rightarrow \mathbb{R}$.

Proof. For example, [2], Theorem 35.1, or see almost any text book in general topology. □

Tietze extension theorem has the following consequence that will be applied to prove the Jordan separation theorem:

Proposition 24.3. Let $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ be closed subsets and let $f: A \rightarrow B$ be a homeomorphism. Then there exists a homeomorphism of pairs

$$F: (\mathbb{R}^m \times \mathbb{R}^n, A \times \{0\}) \rightarrow (\mathbb{R}^n \times \mathbb{R}^m, B \times \{0\}),$$

such that $F(a, 0) = (f(a), 0)$, for every $a \in A$.

Proof. Apply Tietze extension theorem to the coordinate functions of $f: A \rightarrow B \subset \mathbb{R}^n$. This gives a continuous extension $\varphi: \mathbb{R}^m \rightarrow \mathbb{R}^n$ for f . Define the maps

$$\varphi_+: \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^m \times \mathbb{R}^n, (x, y) \mapsto (x, y + \varphi(x)),$$

$$\varphi_-: \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^m \times \mathbb{R}^n, (x, y) \mapsto (x, y - \varphi(x)).$$

Clearly, φ_+ and φ_- are continuous, and it is easy to check that φ_+ is a homeomorphism with the inverse function φ_- ($\varphi_- \circ \varphi_+ = \text{id}_{\mathbb{R}^m \times \mathbb{R}^n} = \varphi_+ \circ \varphi_-$). Let

$$\text{Gr}(f) = \{(a, f(a)) \mid a \in A\}$$

be the graph of f . Then

$$\varphi_+|: A \times \{0\} \rightarrow \mathbb{R}^m \times \mathbb{R}^n, (a, 0) \mapsto (a, 0 + \varphi(a)) = (a, f(a)) \in \text{Gr}(f).$$

The image $\varphi_+(A \times \{0\}) = \text{Gr}(f)$ and it follows that φ_+ takes $A \times \{0\}$ homeomorphically to $\text{Gr}(f)$.

Let $g: B \rightarrow A$ be the inverse function of f , and let $\psi: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a continuous extension of g (such an extension exists by the Tietze extension theorem). As above, there are homeomorphisms

$$\psi_+: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \times \mathbb{R}^m, (y, x) \mapsto (y, x + \psi(y)),$$

$$\psi_-: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \times \mathbb{R}^m, (y, x) \mapsto (y, x - \psi(y)).$$

Then

$$F = \psi_- \circ \tau \circ \varphi_+: \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^m$$

is a homeomorphism, where

$$\tau: \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^m, (x, y) \mapsto (y, x).$$

The homeomorphism F has the desired properties:

1.

$$\begin{aligned} F(a, 0) &= (\psi_- \circ \tau \circ \varphi_+)(a, 0) = (\psi_- \circ \tau)(a, f(a)) \\ &= \psi_-(f(a), a) = (f(a), a - \psi(f(a))) \\ &= (f(a), a - a) = (f(a), 0). \end{aligned}$$

2.

$$A \times \{0\} \xrightarrow{\varphi_+} \text{Gr}(f) \xrightarrow{\tau} \{(f(a), a) \mid a \in A\} \xrightarrow{\psi_-} \{(f(a), 0) \mid a \in A\} = B \times \{0\}.$$

So, F takes $A \times \{0\}$ homeomorphically to $B \times \{0\}$. $\square$

To prove the Jordan separation theorem we will also apply the following result from singular homology:

Theorem 24.4. *Let X be a topological space. Let $A \subset X$ and let $A' \subset A$. Then there is an exact sequence*

$$\cdots \longrightarrow H_n(A, A') \longrightarrow H_n(X, A') \longrightarrow H_n(X, A) \xrightarrow{\Delta} H_{n-1}(A, A') \longrightarrow \cdots$$

Moreover, if there is a commutative diagram $(\odot)$ of pairs of spaces, where the horizontal maps are inclusions,

$$\begin{array}{ccccc} (A, A') & \longrightarrow & (X, A') & \longrightarrow & (X, A) \\ \downarrow & & \downarrow & & \downarrow \\ (B, B') & \longrightarrow & (Y, B') & \longrightarrow & (Y, B), \end{array}$$

then there is a commutative diagram of exact rows:

$$\begin{array}{ccccccc} \cdots \longrightarrow & H_n(A, A') & \longrightarrow & H_n(X, A') & \longrightarrow & H_n(X, A) & \xrightarrow{\Delta} H_{n-1}(A, A') \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & \downarrow \\ \cdots \longrightarrow & H_n(B, B') & \longrightarrow & H_n(Y, B') & \longrightarrow & H_n(Y, B) & \xrightarrow{\Delta'} H_{n-1}(B, B') \longrightarrow \cdots \end{array}$$

where Δ and Δ' are connecting homomorphisms.

Proof. There are short exact sequences of chain complexes:

$$0 \longrightarrow S_*(A)/S_*(A') \longrightarrow S_*(X)/S_*(A') \longrightarrow S_*(X)/S_*(A) \longrightarrow 0$$

and

$$0 \longrightarrow S_*(B)/S_*(B') \longrightarrow S_*(Y)/S_*(B') \longrightarrow S_*(Y)/S_*(B) \longrightarrow 0.$$

The vertical maps in $(\odot)$ induce chain maps that make the following diagram commute:

$$\begin{array}{ccccccc} 0 & \longrightarrow & S_*(A)/S_*(A') & \longrightarrow & S_*(X)/S_*(A') & \longrightarrow & S_*(X)/S_*(A) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & S_*(B)/S_*(B') & \longrightarrow & S_*(Y)/S_*(B') & \longrightarrow & S_*(Y)/S_*(B) \longrightarrow 0 \end{array}$$

It now follows from Corollary 15.13, that there is the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_n(A, A') & \longrightarrow & H_n(X, A') & \longrightarrow & H_n(X, A) \xrightarrow{\Delta} H_{n-1}(A, A') \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & H_n(B, B') & \longrightarrow & H_n(Y, B') & \longrightarrow & H_n(Y, B) \xrightarrow{\Delta'} H_{n-1}(B, B') \longrightarrow \cdots \end{array}$$

□

We will also need a couple of results concerning singular homology of subsets of euclidean spaces:

Proposition 24.5. *Let $A \subset \mathbb{R}^n$ be a closed subset. Then the homology groups*

$$H_k(\mathbb{R}^n, \mathbb{R}^n \setminus A) \text{ and } H_{k+1}(\mathbb{R}^n \times \mathbb{R}, (\mathbb{R}^n \times \mathbb{R}) \setminus (A \times \{0\}))$$

are isomorphic, for every $k \geq 0$.

Proof. Consider the following open subsets of $\mathbb{R}^{n+1}$:

- (1) $H_+ = ((\mathbb{R}^n \setminus A) \times]-1, \infty[) \cup (A \times]0, \infty[)$,
- (2) $H_- = ((\mathbb{R}^n \setminus A) \times]-\infty, 1[) \cup (A \times]-\infty, 0[)$,
- (3) $H_+ \cup H_- = ((\mathbb{R}^n \setminus A) \times \mathbb{R}) \cup (A \times (\mathbb{R} \setminus \{0\})) = \mathbb{R}^{n+1} \setminus (A \times \{0\})$,
- (4) $H_+ \cap H_- = ((\mathbb{R}^n \setminus A) \times]-1, 1[) \cup (A \times \emptyset) = (\mathbb{R}^n \setminus A) \times]-1, 1[$.

We obtain the following diagram:

$$\begin{array}{ccc} H_k(\mathbb{R}^n, \mathbb{R}^n \setminus A) & & H_{k+1}(\mathbb{R}^{n+1}, \overbrace{\mathbb{R}^{n+1} \setminus (A \times \{0\})}^{H_+ \cup H_-}) \\ \gamma \downarrow & & \downarrow \Delta \\ H_k(\mathbb{R}^n, H_+ \cap H_-) & \xleftarrow{\beta} H_k(H_+, H_+ \cap H_-) & \xrightarrow{\alpha} H_k(H_+ \cup H_-, H_-) \end{array}$$

where Δ is the connecting homomorphism of the triple $(\mathbb{R}^{n+1}, H_+ \cup H_-, H_-)$ while α , β and γ are induced by the inclusions

$$(H_+, H_+ \cap H_-) \hookrightarrow (H_+ \cup H_-, H_-),$$

$$(H_+, H_+ \cap H_-) \hookrightarrow (\mathbb{R}^{n+1}, H_+ \cap H_-)$$

and

$$(\mathbb{R}^n, \mathbb{R}^n \setminus A) \cong (\mathbb{R}^n \times \{0\}, (\mathbb{R}^n \setminus A) \times \{0\}) \hookrightarrow (\mathbb{R}^{n+1}, H_+ \cap H_-),$$

respectively. We show that Δ , α , β and γ are isomorphisms.

1. Δ is an isomorphism: Notice that H_- is contractible. Moreover, $x_0 = (b, -1)$, where $b \in \mathbb{R}^n \setminus A$, is a deformation retract of H_- . Therefore,

$$H_k(\mathbb{R}^{n+1}, H_-) \cong H_k(\mathbb{R}^{n+1}, x_0) \cong \tilde{H}_k(\mathbb{R}^{n+1}) = 0,$$

for every k . The long exact sequence of the triple $(\mathbb{R}^{n+1}, H_+ \cup H_-, H_-)$ is

$$\begin{aligned} \cdots \longrightarrow H_{k+1}(H_+ \cup H_-, H_-) &\longrightarrow \underbrace{H_{k+1}(\mathbb{R}^{n+1}, H_-)}_0 \longrightarrow H_{k+1}(\mathbb{R}^{n+1}, H_+ \cup H_-) \\ &\xrightarrow{\Delta} H_k(H_+ \cup H_-, H_-) \longrightarrow \underbrace{H_k(\mathbb{R}^{n+1}, H_-)}_0 \longrightarrow H_k(\mathbb{R}^{n+1}, H_+ \cup H_-) \longrightarrow \cdots \end{aligned}$$

Therefore, Δ is an isomorphism.

2. α is an isomorphism: The inclusion

$$(H_+, H_+ \cap H_-) \hookrightarrow (H_+ \cup H_-, H_-)$$

induces isomorphisms

$$\alpha: H_k(H_+, H_+ \cap H_-) \rightarrow H_k(H_+ \cup H_-, H_-)$$

by Excision 2.

3. β is an isomorphism: The inclusion

$$(H_+, H_+ \cap H_-) \hookrightarrow (\mathbb{R}^{n+1}, H_+ \cap H_-)$$

induces the following commutative diagram, where the rows are exact and the vertical homomorphisms are induced by inclusions:

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & H_{k+1}(H_+ \cap H_-) & \longrightarrow & H_{k+1}(H_+) & \longrightarrow & H_{k+1}(H_+, H_+ \cap H_-) & \longrightarrow & H_k(H_+ \cap H_-) & \longrightarrow & H_k(H_+) & \longrightarrow \cdots \\ & \cong \downarrow & & \cong \downarrow & & \downarrow & & \cong \downarrow & & \cong \downarrow & \\ \cdots \longrightarrow & H_{k+1}(H_+ \cap H_-) & \longrightarrow & H_{k+1}(\mathbb{R}^{n+1}) & \longrightarrow & H_{k+1}(\mathbb{R}^{n+1}, H_+ \cap H_-) & \longrightarrow & H_k(H_+ \cap H_-) & \longrightarrow & H_k(\mathbb{R}^{n+1}) & \longrightarrow \cdots \end{array}$$

Here the homomorphism $H_*(H_+ \cap H_-) \rightarrow H_*(H_+ \cap H_-)$ is induced by the identity map of $H_+ \cap H_-$, hence it is an isomorphism.

For $k > 0$, $H_k(H_+) = 0 = H_k(\mathbb{R}^{n+1})$, and thus the corresponding vertical homomorphism is an isomorphism.

For $k = 0$, $H_0(H_+) \cong \mathbb{Z} \cong H_0(\mathbb{R}^{n+1})$, and the inclusion $H_+ \hookrightarrow \mathbb{R}^{n+1}$ induces a surjective homomorphism $\mathbb{Z} \rightarrow \mathbb{Z}$, hence an isomorphism.

Each row in the above ladder remains exact if we extend the row by adding trivial groups:

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & H_0(H_+ \cap H_-) & \longrightarrow & H_0(H_+) & \longrightarrow & H_0(H_+, H_+ \cap H_-) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots \\ & \cong \downarrow & & \cong \downarrow & & \downarrow & & \cong \downarrow & & \cong \downarrow & \\ \cdots \longrightarrow & H_0(H_+ \cap H_-) & \longrightarrow & H_0(\mathbb{R}^{n+1}) & \longrightarrow & H_0(\mathbb{R}^{n+1}, H_+ \cap H_-) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots \end{array}$$

It now follows from the 5-lemma, that

$$\beta: H_k(H_+, H_+ \cap H_-) \rightarrow H_k(\mathbb{R}^{n+1}, H_+ \cap H_-)$$

is an isomorphism for every $k \geq 0$.

4. γ is an isomorphism: The inclusion

$$(\mathbb{R}^n, \mathbb{R}^n \setminus A) \hookrightarrow (\mathbb{R}^{n+1}, H_+ \cap H_-),$$

where $H_+ \cap H_- = (\mathbb{R}^n \setminus A) \times]-1, 1[$, induces isomorphisms

$$H_k(\mathbb{R}^n) \rightarrow H_k(\mathbb{R}^{n+1})$$

and

$$H_k(\mathbb{R}^n \setminus A) \rightarrow H_k(H_+ \cap H_-),$$

for every $k \geq 0$. An argument similar to the one in case 3 shows that

$$\gamma: H_k(\mathbb{R}^n, \mathbb{R}^n \setminus A) \rightarrow H_k(\mathbb{R}^{n+1}, (\mathbb{R}^n \setminus A) \times]-1, 1[)$$

is an isomorphism for every $k \geq 0$.

Since α , β , γ and Δ are isomorphisms, we obtain an isomorphism

$$\Delta^{-1} \circ \alpha \circ \beta^{-1} \circ \gamma: H_k(\mathbb{R}^n, \mathbb{R}^n \setminus A) \rightarrow H_k(\mathbb{R}^{n+1}, H_+ \cup H_-),$$

for every $k \geq 0$. □

Theorem 24.6. (*Duality theorem*) Let A and B be closed homeomorphic subsets of $\mathbb{R}^n$. Then the groups

$$H_k(\mathbb{R}^n, \mathbb{R}^n \setminus A) \text{ and } H_k(\mathbb{R}^n, \mathbb{R}^n \setminus B)$$

are isomorphic for all $k \geq 0$.

Proof. Let $f: A \rightarrow B$ be a homeomorphism. By Proposition 24.3, there is a homeomorphism of pairs

$$F: (\mathbb{R}^n \times \mathbb{R}^n, A \times \{0\}) \rightarrow (\mathbb{R}^n \times \mathbb{R}^n, B \times \{0\})$$

such that $F(a, 0) = (f(a), 0)$, for every $a \in A$. Then F induces isomorphisms in homology, in particular isomorphisms

$$H_{k+n}(\mathbb{R}^n \times \mathbb{R}^n, (\mathbb{R}^n \times \mathbb{R}^n) \setminus (A \times \{0\})) \rightarrow H_{k+n}(\mathbb{R}^n \times \mathbb{R}^n, (\mathbb{R}^n \times \mathbb{R}^n) \setminus (B \times \{0\})).$$

(Notice here that F maps $(\mathbb{R}^n \times \mathbb{R}^n) \setminus (A \times \{0\})$ homeomorphically to $(\mathbb{R}^n \times \mathbb{R}^n) \setminus (B \times \{0\})$.) Applying Proposition 24.5 n times yields isomorphisms

$$\begin{aligned} H_k(\mathbb{R}^n, \mathbb{R}^n \setminus A) &\cong H_{k+1}(\mathbb{R}^n \times \mathbb{R}, (\mathbb{R}^n \times \mathbb{R}) \setminus (A \times \{0\})) \\ &\cong H_{k+2}(\mathbb{R}^n \times \mathbb{R}^2, (\mathbb{R}^n \times \mathbb{R}^2) \setminus (A \times \{0\})) \\ &\dots \cong H_{k+n}(\mathbb{R}^n \times \mathbb{R}^n, (\mathbb{R}^n \times \mathbb{R}^n) \setminus (A \times \{0\})). \end{aligned}$$

Similarly

$$H_k(\mathbb{R}^n, \mathbb{R}^n \setminus B) \cong H_{k+n}(\mathbb{R}^n \times \mathbb{R}^n, (\mathbb{R}^n \times \mathbb{R}^n) \setminus (B \times \{0\})).$$

Therefore, there are isomorphisms

$$H_k(\mathbb{R}^n, \mathbb{R}^n \setminus A) \cong H_k(\mathbb{R}^n, \mathbb{R}^n \setminus B),$$

for all $k \geq 0$. □

Theorem 24.7. (*Component theorem*) Let A and B be closed homeomorphic subsets of $\mathbb{R}^n$. Then $\pi_0(\mathbb{R}^n \setminus A)$ and $\pi_0(\mathbb{R}^n \setminus B)$ have the same cardinality.

Proof. Assume first that $A \neq \mathbb{R}^n$ and that $B \neq \mathbb{R}^n$. The long exact homotopy sequence of the pair $(\mathbb{R}^n, \mathbb{R}^n \setminus A)$ ends as follows

$$\begin{aligned} \cdots \longrightarrow \underbrace{H_1(\mathbb{R}^n)}_0 \longrightarrow H_1(\mathbb{R}^n, \mathbb{R}^n \setminus A) \longrightarrow H_0(\mathbb{R}^n \setminus A) \\ \xrightarrow{f} \underbrace{H_0(\mathbb{R}^n)}_{\cong \mathbb{Z}} \longrightarrow \underbrace{H_0(\mathbb{R}^n, \mathbb{R}^n \setminus A)}_0. \end{aligned}$$

Then f is a surjection. Let $x \in H_0(\mathbb{R}^n \setminus A)$ be such that $f(x) = 1$ equals a generator of $H_0(\mathbb{R}^n)$. Define $g: H_0(\mathbb{R}^n) \rightarrow H_0(\mathbb{R}^n \setminus A)$ by $g(1) = x$ and $g(n) = nx = x + \cdots + x$. Then g is a homomorphism and $f \circ g = \text{id}: \mathbb{Z} \rightarrow \mathbb{Z}$. Therefore, the sequence above splits, i.e.,

$$H_0(\mathbb{R}^n \setminus A) \cong H_1(\mathbb{R}^n, \mathbb{R}^n \setminus A) \oplus \mathbb{Z}.$$

Now, $H_0(\mathbb{R}^n \setminus A)$ is a free abelian group whose *rank* (= the cardinality of a basis) equals the cardinality of $\pi_0(\mathbb{R}^n \setminus A)$. Similarly,

$$H_0(\mathbb{R}^n \setminus B) \cong H_1(\mathbb{R}^n, \mathbb{R}^n \setminus B) \oplus \mathbb{Z}.$$

By Theorem 24.6,

$$H_1(\mathbb{R}^n, \mathbb{R}^n \setminus A) \cong H_1(\mathbb{R}^n, \mathbb{R}^n \setminus B).$$

Thus

$$H_0(\mathbb{R}^n \setminus A) \cong H_0(\mathbb{R}^n \setminus B),$$

and it follows that $\pi_0(\mathbb{R}^n \setminus A)$ and $\pi_0(\mathbb{R}^n \setminus B)$ have the same cardinality.

If $A = \mathbb{R}^n$, then by the invariance of domain (to be proved soon!) also B is open. Since B is also closed and $\mathbb{R}^n$ is connected, it follows that also $B = \mathbb{R}^n$. $\square$

Remark 24.8. We will use Theorem 24.7 to prove results that are used to prove the invariance of domain. However, we will only use the case where $A, B \neq \mathbb{R}^n$. Thus we are not using the invariance of domain to prove the invariance of domain!

Let $f: \mathbb{S}^1 \hookrightarrow \mathbb{R}^2$ be an injective continuous mapping. Since $\mathbb{S}^1$ is compact and $\mathbb{R}^2$ is Hausdorff, it follows that f is a closed map. Therefore, f is an *embedding*, i.e., f is a homeomorphism onto the image $f(\mathbb{S}^1)$.

Theorem 24.9. (*Jordan separation theorem*) Let $S \subset \mathbb{R}^n$ be homeomorphic to the sphere $\mathbb{S}^{n-1}$, where $n \geq 2$. Then $\mathbb{R}^n \setminus S$ has two path components: a bounded path component J ("interior") and an unbounded path component A ("exterior"). Moreover, S is the set of boundary points of J and A . (Thus J and A do not contain their boundary points, which means that they are open in $\mathbb{R}^n$.)

Proof. The claim is true if $S = \mathbb{S}^{n-1}$. It now follows from Theorem 24.7 that the complement $\mathbb{R}^n \setminus S$ has two path components. Clearly, at least one of the path components is unbounded, denote that path component by A . If both path components were unbounded, then also S would have to be so. Therefore, the other path component J is bounded.

Let's next consider the boundary points of J and A . Let $x \in S$, and let V be an open neighborhood of x in $\mathbb{R}^n$. Then $C = S \setminus (S \cap V)$ is closed in S and homeomorphic to a closed subset of $\mathbb{S}^{n-1}$. Denote this subset by D , notice that

$D \neq \mathbb{S}^{n-1}$. Then $\mathbb{R}^n \setminus D$ is path-connected (there is $x \in \mathbb{S}^{n-1} \cap (\mathbb{R}^n \setminus D)$, if $y \in \text{int} D^n$ and $z \in \mathbb{R}^n \setminus D$, then there is a path $y \rightarrow z$ in $\mathbb{R}^n \setminus D$ going through x .) Theorem 24.7 implies that $\mathbb{R}^n \setminus C$ is path-connected.

Let $p \in J$, $q \in A$, and $w: [0, 1] \rightarrow \mathbb{R}^n \setminus C$ be a path from p to q . Then $w^{-1}(S) \neq \emptyset$. Since $w^{-1}(S)$ is compact, it has a minimum element t_1 and a maximum element t_2 . Then $w(t_1), w(t_2) \in S \cap V$. Then $w([0, t_1]) \subset \mathbb{R}^n \setminus S$. Since $w([0, t_1])$ is path-connected and $p = w(0) \in J$, it follows that $w([0, t_1]) \subset J$. Similarly, $w((t_2, 1]) \subset A$. Now, $w(t_1)$ is a limit point (= an accumulation point) of $w((t_2, 1])$. Hence there is $t_3 \in [0, t_1)$ with $w(t_3) \in J \cap V$ and $t_4 \in (t_2, 1]$ with $w(t_4) \in A \cap V$. It follows that x is contained in the boundary of J and in the boundary of A . $\square$

Remark 24.10. The separation theorem can be improved for $n = 2$. The *Schoenflies theorem* says the following: Let $S \subset \mathbb{R}^2$ be a Jordan curve. Then there is a homeomorphism $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $f(S) = \mathbb{S}^1$.

Theorem 24.11. *Let $A \subset \mathbb{R}^n$, $n > 1$, be homeomorphic to the closed k -dimensional unit disk $\mathbb{D}^k$, $k \leq n$. Then $\mathbb{R}^n \setminus A$ is path-connected.*

Proof. Since $\mathbb{D}^k$ is compact, it follows that A is compact, which implies that A is closed in $\mathbb{R}^n$. Since $\mathbb{R}^n \setminus \mathbb{D}^k$ is path-connected, Theorem 24.7 implies that $\mathbb{R}^n \setminus A$ is path-connected. $\square$

Theorem 24.12. (*Invariance of domain*) *Let $U \subset \mathbb{R}^n$ be open, and let $f: U \rightarrow \mathbb{R}^n$ be an injective continuous map. Then $f(U)$ is open in $\mathbb{R}^n$, and f maps U homeomorphically onto $f(U)$. Let $V \subset \mathbb{R}^n$ be homeomorphic to an open subset U of $\mathbb{R}^n$. Then V is open in $\mathbb{R}^n$.*

Proof. 1. The case $n = 1$ is left as an exercise.

2. Assume then $n \geq 2$. We show that f is an open map. It then follows that $f(U)$ is open and that f takes U homeomorphically onto $f(U)$. Let $a \in U$ and let $\delta > 0$ be so small that

$$D = \{x \in \mathbb{R}^n \mid \|x - a\| \leq \delta\} \subset U.$$

Let S be the boundary of D . It suffices to show that $f(\mathring{D})$ is open, where

$$\mathring{D} = \{x \in \mathbb{R}^n \mid \|x - a\| < \delta\}$$

is the interior of D . Both S and $T = f(S)$ are homeomorphic to $\mathbb{S}^{n-1}$. Theorem 24.7 implies that $\mathbb{R}^n \setminus T$ has two path components, U_1 and U_2 . By Theorem 24.9, one of the path components is bounded and the other is unbounded. Let U_2 be the unbounded one. It follows from Theorem 24.11 that $\mathbb{R}^n \setminus f(D)$ is path-connected. Since $\mathbb{R}^n \setminus f(D) \subset \mathbb{R}^n \setminus T$, it follows that $\mathbb{R}^n \setminus f(D)$ is contained in U_1 or in U_2 . Since $f(D)$ is compact, $\mathbb{R}^n \setminus f(D)$ is unbounded, and thus $\mathbb{R}^n \setminus f(D) \subset U_2$. Thus $T \cup U_1 = \mathbb{R}^n \setminus U_2 \subset f(D)$ and $U_1 \subset f(D) \setminus T = f(\mathring{D})$. Since $\mathring{D}$ is path-connected, also $f(\mathring{D})$ is path-connected. Since $f(\mathring{D}) \subset \mathbb{R}^n \setminus T = U_1 \cup U_2$, must be $f(\mathring{D}) \subset U_1$. Thus $f(\mathring{D}) = U_1$ is open in $\mathbb{R}^n$. It follows that f is an open map.

The second statement follows immediately from the first one. $\square$

Theorem 24.13. (*Invariance of dimension*) Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open subsets, $U, V \neq \emptyset$. Assume U and V are homeomorphic. Then $m = n$.

Proof. Assume $m < n$. Let $h: U \rightarrow V$ be a homeomorphism, and let $i: \mathbb{R}^m \rightarrow \mathbb{R}^n$, $x \mapsto (x, 0, \dots, 0)$. Then

$$f: V \rightarrow \mathbb{R}^n, \quad y \mapsto h^{-1}(y) \mapsto (h^{-1}(y), 0, \dots, 0),$$

is a continuous injection. Invariance of domain implies that $f(V)$ is open in $\mathbb{R}^n$, which is a contradiction. $\square$

25. APPENDIX: FREE ABELIAN GROUPS

Theorem 25.1. Let F be a free abelian group, and let B and C be abelian groups. Let $g: B \rightarrow C$ be a surjective homomorphism. Let $h: F \rightarrow C$ be a homomorphism. Then there exists a homomorphism $f: F \rightarrow B$ with $g \circ f = h$.

$$\begin{array}{ccc} & F & \\ & \downarrow h & \\ B & \xrightarrow{g} & C \longrightarrow 0 \\ & \nwarrow f & \end{array}$$

Proof. Let X be a basis for F . Since g is surjective, it follows that for every $x \in X$ there is an element $b_x \in B$ with $g(b_x) = h(x)$. The function $x \mapsto b_x$ defines a homomorphism $f: F \rightarrow B$ (extend $x \mapsto b_x$ by linearity). For every $x \in X$,

$$g(f(x)) = g(b_x) = h(x).$$

Since $(g \circ f)(x) = h(x)$ for the basis elements $x \in X$, it follows that $g \circ f = h$. $\square$

Corollary 25.2. Let F be a free abelian group, and let B be an abelian group. Let $g: B \rightarrow F$ be a surjective homomorphism. Then $B = \ker g \oplus F'$, where $F' \cong F$.

Proof. Consider the diagram

$$\begin{array}{ccc} & F & \\ & \downarrow \text{id} & \\ B & \xrightarrow{g} & F \longrightarrow 0 \\ & \nwarrow f & \end{array}$$

By Theorem 25.1, there is a homomorphism $f: F \rightarrow B$ with $g \circ f = \text{id}$. Thus f is injective. Check that $B = \ker g \oplus \text{im } f$: Let $b \in B$. Then

$$b = b - f(g(b)) + f(g(b)),$$

where $g(b) \in F$, $f(g(b)) \in \text{im } f$, and

$$g(b - f(g(b))) = g(b) - \underbrace{(g \circ f)}_{\text{id}}(g(b)) = 0.$$

Thus $b - f(g(b)) \in \ker g$.

Assume $b \in \ker g \cap \text{im } f$. Then $b = f(x)$, for some $x \in F$, and

$$0 = g(b) = g(f(x)) = (g \circ f)(x) = x.$$

Thus $b = f(x) = f(0) = 0$. It follows that $\ker g \cap \operatorname{im} f = \{0\}$. Thus

$$B = \ker g \oplus \operatorname{im} f,$$

where $F' = \operatorname{im} f \cong F$. □

Definition 25.3. Let A, B, C be abelian groups. An exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$

is called *split* (or a *split exact sequence*), if there is a homomorphism $s: C \rightarrow B$ with $ps = \operatorname{id}_C$.

Lemma 25.4. *The following statements are equivalent:*

(1) *The exact sequence*

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$

is split.

(2) *There is a homomorphism $q: B \rightarrow A$ with $qi = \operatorname{id}_A$.*

(3) *The group A is a direct summand of B (that is, there exists a subgroup C' of B such that the restriction $p|_C: C' \rightarrow C$ is an isomorphism and $B = \operatorname{im}(i) \oplus C'$).*

Proof. Exercise. □

Theorem 25.5. *Let F be a free abelian group. Let H be a subgroup of F . Then H is free abelian and $\operatorname{rank}(H) \leq \operatorname{rank}(F)$. In particular, H is finitely generated if F is finitely generated.*

Proof. We give two proofs. The first one only works for F having finite rank.

1. Assume F has finite rank n . We prove the claim by induction on n .

Let $n = 1$. Then $F \cong \mathbb{Z}$. Any subgroup of $\mathbb{Z}$ is either 0 or isomorphic to $\mathbb{Z}$. Thus H is free abelian and $\operatorname{rank}(H) \leq 1 = \operatorname{rank}(F)$.

Assume then F has basis $\{x_1, \dots, x_n\}$. Let $F_n = \langle x_1, \dots, x_{n-1} \rangle$, and let $H_n = H \cap F_n$. By induction, H_n is a free abelian group, and $\operatorname{rank}(H_n) \leq n - 1$. Now,

$$H/H_n = H/(H \cap F_n) \cong (H + F_n)/F_n \subset F/F_n \cong \mathbb{Z}.$$

Thus H/H_n is isomorphic to a subgroup of $\mathbb{Z}$. If $H/H_n = 0$, then $H = H_n$. If $H/H_n \cong \mathbb{Z}$, then it follows from Corollary 25.2 that $H = H_n \oplus \langle y_n \rangle$, where $\langle y_n \rangle \cong \mathbb{Z}$. Thus H is a free abelian group of rank less than equal to $(n - 1) + 1 = n$. This completes the proof where $\operatorname{rank}(F)$ is finite.

2. Assume then that the rank of F is not necessarily finite. Let $\{x_k \mid k \in K\}$ be a basis of F . We assume $\{x_k \mid k \in K\}$ is well-ordered. (That every non-empty set can be well-ordered is equivalent to the axiom of choice.) For $k \in K$, let

$$F_k = \sum_{j < k} \langle x_j \rangle, \quad \bar{F}_k = \sum_{j \leq k} \langle x_j \rangle,$$

$$H_k = H \cap F_k, \quad \bar{H}_k = H \cap \bar{F}_k.$$

Now,

$$F = \bigcup_k \bar{F}_k, \quad H = \bigcup_k \bar{H}_k,$$

and

$$H_k = H \cap F_k = H \cap F_k \cap \bar{F}_k = \bar{H}_k \cap F_k.$$

Thus

$$\begin{aligned} \bar{H}_k/H_k &= \bar{H}_k/(\bar{H}_k \cap F_k) \cong (\bar{H}_k + F_k)/F_k \\ &\subset \bar{F}_k/F_k \cong \mathbb{Z}. \end{aligned}$$

As in the first proof, either $\bar{H}_k = H_k$, or $\bar{H}_k = H_k \oplus \langle h_k \rangle$, where $\langle h_k \rangle \cong \mathbb{Z}$.

We show that H is a free abelian group with basis $\{h_k\}$. It will then follow that

$$\text{rank}(H) = \text{card}\{h_k\} \leq \text{card}(K) = \text{rank}(F).$$

Let H^0 be the subgroup of H generated by the h_k . Now, each $h \in H$ lies in some $\bar{F}_k$, since F is the union of the $\bar{F}_k$. Let $\mu(h)$ be the least index k with $h \in \bar{F}_k$. Assume $H \neq H^0$. Consider the set

$$A = \{\mu(h) \mid h \in H, h \notin H^0\}.$$

Since K is well-ordered, it follows that A has a least index j . Let $h' \in H$ be such that $\mu(h') = j$ and $h' \notin H^0$. Then $\mu(h') = j$ implies that $h' \in H \cap \bar{F}_j$. Thus $h' = a + mh_j$, where $a \in H_j$ and $m \in \mathbb{Z}$. Thus $a = h' - mh_j \in H$, $a \notin H^0$ and $a \in H_j$. It follows that $\mu(a) < j$, which is a contradiction. Therefore, $H = H^0$.

It remains to show that the linear combinations of the h_k are unique. It suffices to show that if

$$m_1 h_{k_1} + \cdots + m_n h_{k_n} = 0, \quad k_1 < \cdots < k_n,$$

then $m_1 = \cdots = m_n = 0$. Assume $m_n \neq 0$. Then

$$m_n h_{k_n} = -m_1 h_{k_1} - \cdots - m_{n-1} h_{k_{n-1}} \in \langle h_{k_n} \rangle \cap H_{k_n} = 0,$$

which is a contradiction. Thus the linear combinations of the h_k are unique, and it follows that H is a free abelian group with basis $\{h_k\}$. □

26. ENGLISH-FINNISH DICTIONARY

Here comes a list of some words that are frequently used in topology. Feel free to suggest better Finnish words!

boundary	reuna
boundary homomorphism	reunahomomorfismi
category	kategoria
chain	ketju
chain complex	ketjukompleksi
class	luokka
coboundary	koreuna
cochain	koketju
cocycle	kosykli
cochain complex	koketjukompleksi
cohomology	kohomologia
connected	yhtenäinen
connecting homomorphism	yhdistävä homomorfismi
contractible	kutistuva
cup product	kuppitulo
cycle	sykli
deformation retract	deformaatioretrakti
direct product	suora tulo
direct sum	suora summa
exact sequence	eksakti jono
excision	typistys
diagram	kaavio
face	tahko
free abelian group	vapaa Abelin ryhmä
functor	funktori
fundamental group	perusryhmä
graded ring	porrasteinen rengas
groupoid	grupoidi
homology	homologia
homotopy	homotopia
loop	silmutta
morphism	morfismi
natural transformation	luonnollinen transformaatio
nullhomotopy	nollahomotopia
object	objekti
path	polku
path component	polkukomponentti
path-connected	polkuyhtenäinen
pullback	nykäisy
pushout	töytäisy
retract	retrakti
retraction	retraktio
simplex	simpleksi
smash	nitistys
split exact sequence	halkeava eksakti jono
suspension	suspensio

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