Chapter 11
Husserl and Hilbert

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Abstract The paper examines Husserl's (1859–1938) phenomenology and Hilbert's (1862–1943) view of the foundations of mathematics against the backdrop of their lifelong friendship. After a brief account of the complementary nature of their early approaches, the paper focuses on Husserl's *Formale und transzendentale Logik* (1929) viewed as a response to Hilbert's "new foundations" developed in the 1920s. While both Husserl and Hilbert share a "mathematics first," nonrevisionist approach toward mathematics, they disagree about the way in which the access to it should be construed: Hilbert wanted to reach it and show it consistent by his formalism on the basis of sensuous signs, Husserl held that there should be a reduction to elementary judgements about individuals. Husserl's reduction does not establish the consistency of mathematics but he claims it is important for the considerations of truth.

Keywords Edmund Husserl • Foundations of mathematics • Hilbert's program • Dietrich Mahnke • Formalism

11.1 Husserl's and Hilbert's Friendship

Since 1901, from his arrival to Göttingen, until he left for Freiburg in 1916, Husserl's closest colleague and the main defender at the university seems to have been the mathematician David Hilbert. While Husserl befriended the mathematicians Felix Klein and especially Hilbert, the philosophers' attitude towards Husserl was sultry if not altogether hostile throughout his stay in Göttingen. The philosophical faculty had opposed Husserl's appointment initially in 1900 when the Prussian Ministry of Culture first expressed such an intention. The faculty had hoped

I wish to thank Øystein Linnebo, Matti Eklund, Besim Karakadilar, Mitsuhiro Okada, Volker Peckhaus, and Sören Stenlund for valuable feedback on earlier versions of this paper.

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S. Centrone (ed.), *Essays on Husserl's Logic and Philosophy of Mathematics*,
Synthese Library 384, DOI 10.1007/978-94-024-1132-4_11

for expertize in philologico-historical problems, but instead in 1901 they received Husserl whose background in mathematics and notorious anti-psychologism did not make him particularly popular among the psychologists of the department.[1] Husserl's problems continued in 1905 when ministry tried to promote Husserl as an ordinary professor. While the ordinary professors in philosophy, Julius Baumann and Georg Elias Müller objected to promoting Husserl, Hilbert defended Husserl and wrote seven reviews about his scientific achievements to the effect that he was eventually promoted as an "Ordinarius für Philosophie" in 1906. Hilbert promoted Husserl also in 1908 when a position as a "zusätzliches Philosophie-Ordinariat" was opened in Göttingen claiming that it is utmost important to try to keep Husserl in Göttingen. This time the position however was eventually given to Heinrich Maier.[2] The reportedly deep and respectful friendship between Husserl and Hilbert continued throughout their lives.[3] Given their common history, it is no wonder that in a letter for Hilbert's 60th birthday Husserl expressed gratitude for Hilbert's constant interest and trust in him in his philosophical struggles. In his letter Husserl also highlights the importance of Hilbert's mathematical creativity for philosophy.[4]

In this paper my aim is to clarify Husserl's view in respect of Hilbert's achievements in the foundations of mathematics. I will first discuss their views right after the turn of the twentieth century, but my main focus will be on their views in the 1920s. One motivation for this is Husserl's student's Friedrich Mahnke's insistence of the phenomenological nature of Hilbert's program. Accordingly, I will discuss Husserl's *Formale und transzendentale Logik* (1929)[5] as a response to Hilbert's program as developed in the early 1920s. Husserl thinks that a careful phenomenological analysis shows that the consistency of mathematical theories has to be approached semantically, by taking into account "the matters in the synthetic unity of the experience."[6] I will argue that Husserl's approach is model theoretical and for him *in mathematics* there is no need for direct consistency proof sought for by Hilbert. He nevertheless formulated a reduction of mathematics to primitive judgments, which for him shows the relationship of formal mathematics to truth and is important for *philosophical* reasons.

[1]Peckhaus 1990, 2016–208; Husserl Archive Mitteilungsblatt 34, 12, see also Hill & Da Silva 2013.

[2]Peckhaus 1990, 208–210.

[3]The most recent details of Husserl's and Hilbert's friendship and how it also extended to their families have been documented in the Husserl Archive Leuven *Mitteilungsblatt* 36, 2013.

[4]Husserl 1994, 119.

[5]Husserl 1974. Henceforth cited as *FTL*. English translations refer to (Husserl 1969) unless otherwise indicated.

[6]*FTL*, §89b.

11.2 The More General Background in the Development of Mathematics

The change of the century witnessed an emergence of modern abstract view of mathematics. The development towards abstract conceptual viewpoint centered in Göttingen of Felix Klein and David Hilbert where Husserl also moved in 1901.[7] Klein shared Hilbert's understanding of mathematics as a multifaceted but fundamentally unified body of knowledge.[8] While Klein's aim was to place mathematics on a healthy footing by means of his massive *Enzyklopädie der mathematischen Wissenschaften*, a survey in six volumes covering all mathematical branches, including applied mathematics, Hilbert developed the axiomatic approach. He held that[9]

> [m]athematics is an indivisible whole, an organism whose livelihood depends on the interconnections between its parts... The further a mathematical theory is developed, the more harmonious and unified its structure unfolds, leading to the discovery of relationships between previously distinct branches of knowledge. Thus it happens that with the expansion of mathematical knowledge, the holistic quality becomes enhanced rather than lost.

Like Klein and Hilbert Husserl also entertained a picture of a unifying theoretical framework for different mathematical theories. Husserl envisioned "a definite, ordered procedure which will enable us to construct the possible forms of theories, to survey their law-governed connections, and to pass from one to another by varying their basic determining factors, etc".[10] Husserl's theory of theories is a mathematical theory within which individual theories could be examined formally. As Husserl's student Dietrich Mahnke puts it, Husserl emphasized the importance of the "eternally valid systematic structure of formal mathematics as the ideal storeroom of the forms of theories of all exact science that it places with clear consciousness, for the first time since Leibniz, at the entrance to the unified theory of science."[11]

However, the theory of theories was not something a philosopher could impose on mathematicians. "No one can debar mathematicians from staking claims to all that can be treated in terms of mathematical form and method."[12] In his

[7]cf. Ferreirós 2007, 31. Soon after the move to Göttingen, Husserl's wife, Malvine, reported that in Göttingen a ‚ganz anderer Zug im geistigen Leben der Universität als in Halle <herrsche>, u. besonders sind es die Mathematiker (Klein u. Hilbert), die Edmund in ihren Kreis ziehen u. ihn <...> anregen." David Hilbert and Edmund Husserl developed a ‚tiefe achtungsvolle Freundschaft" which, according to Husserl's wife, was a consequence of the ‚gleichen Ethos einer restlosen Hingabe an sein Werk" (Husserl Archive Mitteilungsblatt 2013, 15).

[8]Rowe 1989, 198.

[9]Cited from Rowe 1989, 212.

[10]Husserl 1975, henceforth cited as *Prolegomena*, §69, translation modified.

[11]Mahnke 1977 [1923], 75.

[12]*Prolegomena*, §71.

Prolegomena Husserl advocated a division of labor, according to which mathematicians' task was to be ingenious technicians constructing theories and solving problems, while philosophers' task was to provide ultimate insight to the essence of these mathematical theories.[13] The division of labor reflects Husserl's general approach towards mathematics: mathematics comes first, only afterwards its theories should be examined by a philosopher. Similar, "mathematics-first" approach to the foundations of mathematics can also be found in Hilbert's explanation of how, contrary to construction of buildings, in the house of knowledge the foundations are given only after the comfortable rooms have been built. Eventually, however, for both, Husserl and Hilbert, the examination of the foundations became more seriously motivated by the paradoxes found around the turn of the century. What distinguishes the two is that in accordance to the division of labor, Hilbert (and other mathematicians) focused on freely constructing theories, Husserl saw as his own task to provide understanding for their essence and clarifying fundamental concepts of the constructed theories. He thus held his own approach, not as abandoning or sidestepping mathematics, but as importantly complementing it.

11.3 Hilbert on Foundations

Hilbert's interest in the foundations came in two periods: first in 1899–1905 and later again in 1917 onwards. The first period marks the beginning of Husserl's career in Göttingen until his discovery of phenomenological reduction. During the second period, Hilbert developed his 'Program.' At this point Husserl had already moved to Freiburg. However, he still worked on the foundations of mathematics as is witnessed by his *Formal and Transcendental Logic* (1929).

The purpose of Hilbert's axiomatics was to provide mathematics and physical sciences secure foundations. Hilbert compared his axiomatic method to the genetic method used by Weierstrass, Kronecker, and Dedekind among others. In the genetic method one typically started from the number 1 and then extended the number domain by adding the results of calculation into it. In the axiomatic method, on the contrary, one begins by assuming the existence of all the elements, and then one brings these elements into relationship with one another by means of certain axioms:

> The necessary task then arises of showing the consistency and the completeness of these axioms, i.e. it must be proved that the application of the given axioms can never lead to contradictions, and, further, that the system of axioms is adequate to prove all geometrical propositions. We shall call this procedure of investigation the axiomatic method.

Hilbert continues: "Despite the high pedagogic and heuristic value of the genetic method, for the final presentation and the complete logical grounding [Sicherung] of our knowledge the axiomatic method deserves the first rank."[14] Contrary to the

[13] *Loc. cit.*

[14] Hilbert 1900a, 1092–1093.

earlier genetic approach, with the axiomatic approach it is possible to compress what we know in some field of knowledge into a logical axiom system and thus it enables obtaining an overview of this entire field.

In 1899 Hilbert's axiomatization of geometry, *Grundlagen der Geometrie* (*Foundations of Geometry*) appeared in print. In it Hilbert presented the axioms of Euclidean geometry divided into five groups: axioms of incidence, axioms of order, axioms of congruence, axiom of parallels, and axioms of continuity. Hilbert established the consistency of the axiom system by constructing a model of real numbers that satisfies the axioms. The consistency of geometry is thus established assuming the consistency of real number system, and thus he managed to prove only a relative consistency of geometry. But, similarly, by model-theoretical means, he also established the independence of various axioms from each other.

To ensure that the axiomatic method indeed provides the complete logical grounding of our knowledge Hilbert soon added the so-called axiom of completeness (*Vollständigkeitsaxiom*) to his axiomatization. The axiom appeared in print already in the French translation of the *Grundlagen die Geometrie* as well as in his axiomatization of arithmetic *Über den Zahlbegriff* (1900a). According to Hilbert's Completeness axiom[15]:

> To a system of points, straight lines, and planes, it is impossible to add other elements in such a manner that the system thus generalized shall form a new geometry obeying all of the five groups of axioms. In other words, the elements of geometry form a system which is not susceptible of extension, if we regard the five groups of axioms as valid.

The number theoretical formulation of the same axiom is the following[16]:

> It is not possible to add to the system of numbers another system of things so that the axioms I, II, III, and IV are also all satisfied in the combined system; in short, the numbers form a system of things which is incapable of being extended while continuing to satisfy all the axioms.

With this axiom Hilbert made his system of axioms categorical. A set of axioms is categorical if it has a unique model up to isomorphism. The completeness axiom is thus an axiom about models of axioms. Hilbert however did not have a general notion of isomorphism, nor did he have a precise notion of formal deduction at use, so it is not entirely clear what exactly he meant by "completeness".[17]

The consistency of the axiomatization remained a problem for Hilbert. In "On the Concept of Number" Hilbert thought that "one needs only a suitable modification of familiar methods of inference".[18] This proved to be more difficult than what he first anticipated. Already in the same year Hilbert listed the consistency of the arithmetical axioms as the second problem in his list of 23 unsolved problems in his famous address to the International Congress of Mathematicians in Paris. Hilbert's

[15] Hilbert 1950, 15.

[16] Hilbert 1900a, 1094.

[17] Awodey & Reck 2001, 11–20.

[18] Hilbert 1900a, 1095.

formulation is now more careful: “I am convinced that it must be possible to find a direct proof for the consistency of the arithmetical axioms, by means of a careful study and suitable modification of the known methods of reasoning in the theory of irrational numbers.”[19]

From about 1917 onwards Hilbert, in collaboration with Bernays, gave a series of lectures, in which the modern mathematical logic and proof theory were conceived.[20] The development of the so called ‘Hilbert Program’ was first made public in the lectures given in the winter term 1921/1922.[21] Incidentally, Dietrich Mahnke held that the presentation of Hilbert’s axiomatics in lectures in 1921 comes closest to Husserl’s phenomenological foundation.[22] The basic idea in Hilbert’s ‘Program’ is to turn the propositions that constitute mathematics into formulas. Certain formulas are called axioms, by my means of which further formulas can be derived. “The axioms and provable propositions, that is, the formulas resulting from this procedure, are copies (*Abbilder*) of the thoughts constituting customary mathematics as it has developed till now”.[23] With such formal deductive systems he sought to capture the customary mathematics, his intention was not to restrict it. By means of such approach Hilbert thought he could cover all the infinitistic mathematics and finally prove the consistency of arithmetic by finitary methods.

Hilbert motivates his project philosophically. As a presupposition of his program, Hilbert mentions that[24]

> something must already be given to us in our faculty of representation [in der Vorstellung], certain extralogical concrete objects that are intuitively [anschaulich] present as immediate experience prior to all thought. If logical inference is to be reliable, it must be possible to survey these objects completely in all their parts, and the fact that they occur, that they differ from one another, and that they follow each other, or are concatenated, is immediately given intuitively, together with the objects, as something that neither can be reduced to anything else nor requires reduction.

This, according to Hilbert, is the basic philosophical requirement for mathematics and all scientific thinking. In mathematics, the objects in question are concrete signs themselves, “whose shape … is immediately clear and recognizable”.[25] The aim of Hilbert’s theory is that, in Hilbert’s own words, “contentual inference is replaced by manipulation of signs (*äußeres Handeln*) according to rules; in this way the axiomatic method attains that reliability and perfection that it can and must reach if it is to become the basic instrument of all theoretical research”.[26]

[19]Hilbert 1900b, 1104.
[20]Sieg 1999, 12.
[21]*Op. cit.*, 23, Sieg 2013, 115.
[22]Mahnke 1977, 76–77.
[23]*Op. cit.*, 465.
[24]*Op. cit.*, 464–465.
[25]*Op. cit.*, 465.
[26]*Op. cit.*, 467.

Hilbert's words may suggest an interpretation that to him mathematics *is* "manipulation of signs according to rules," and that Hilbert thus is a dogmatic formalist about mathematics. This however is not true and accordingly in the secondary literature, the view has been repeatedly rejected. David E. Rowe, for example, characterizes Hilbert as the leading proponent of the 'structuralist' tradition in modern mathematics rooted in the axiomatics, Cantor's set theory and the algebraic tradition. According to Rowe, it is a mistake to identify Hilbert's philosophy of mathematics with his formalist program, the purpose of which was to legitimate the entire corpus of mathematical knowledge.[27] Rowe suggests further that, as Herbert Mehrtens has put it, Hilbert's position in the *Grundlagenkrise* was that "Mathematicians can philosophize about mathematics, but they are not permitted to draw normative consequences that would constrain scientific production".[28] Similar view is expressed also by W. Sieg who points out that Hilbert and Bernays always present their theories together with a structure, calling it the 'existential aspect' of the axiomatic method.[29] According to him, Hilbert's program seeks a uniform structural reduction: "intended structures are projected through their assumed complete formalizations into the properly mathematical domain ..., i.e., finitist mathematics."[30] Furthermore, in the spirit of his "mathematics-first" approach, Hilbert's view of the rival approaches to the foundations is that they, too, enhance our understanding of the mathematical content of theories. Thus he viewed Brouwer's and Weyl's, work as part of axiomatic investigations.[31] I will argue soon that something like this is also Husserl's view, namely that Husserl, too, wants to show how abstract mathematics is accessible from the meaningful concrete basis.

11.4 Husserl on the Foundations

As already mentioned above, Hilbert's view of mathematics as a unified but multifaceted axiomatic science seems to be close to what Husserl means by a theory of theories in the *Prolegomena*. The theory of the possible forms of theories deals a priori with the essential forms of theories and the relevant laws of relation. Husserl certainly wants the theories to be axiomatic, or "nomological," that is the term used by Husserl (§64). In 1901 upon arrival at Göttingen Husserl was invited to give two lectures on the notion of definiteness of axiom systems. In those lectures Husserl defines the concepts 'relatively' and 'absolutely definite' axiom systems. The relatively definite axiom system is one to which one cannot add any new axioms without defining a new domain. It thus defines its objects so that they are

[27] Rowe 1989, 199–200.

[28] *Op. cit.*, 211.

[29] Sieg 2013, 106.

[30] *Op. cit.*, 316.

[31] Sieg 2009, 450.

completely unambiguous (i.e., in modern terminology, they do not admit any non-standard interpretations). The absolutely definite axiom system is the one that has the maximal set of axioms.[32] Both are categorical at least in some general sense. The relatively definite axiom system has a unique model, and thus it defines its objects completely unambiguously. An absolutely definite theory is one that cannot be extended consistently (for more detail, see Hartimo 2016). Indeed, in it "the numbers form a system of things which is incapable of being extended while continuing to satisfy all the axioms"[33] as Hilbert defined his completeness axiom.

Husserl himself held that absolute definiteness captures completeness in Hilbert's sense.[34] But Husserl disagreed with Hilbert in holding that completeness should not be viewed as an axiom but as a theorem.[35] Later, in the *Ideen I* Husserl writes that "the close relationship of the concept of definiteness to the 'axiom of completeness' introduced by Hilbert for the foundation of arithmetic will be immediately obvious to every mathematician".[36] In *Formal and Transcendental Logic* (§31) Husserl points out that

> Hilbert arrived at his concept of completeness (naturally quite independently of my [Husserl's] still-unpublished investigations); he attempts, in particular, to complete a system of axioms by adding a separate 'axiom of completeness.' The above-given analyses should make it evident that, even if the inmost motives that guided him mathematically were inexplicit, they tended essentially in the same direction as those that determined the concept of the definite multiplicity.

Husserl thus, also retrospectively, held that like Hilbert, he was advocating categoricity of the axiom systems with his notion of definiteness. Similar interpretation is supported by Dietrich Mahnke's 1923 explanation of Husserlian "theory-forms" as the common deductive scaffolding of all 'logically isomorphic' or 'formally equivalent' disciplines.[37] Obviously, as in Hilbert's case, without the precise notions

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[32]In Husserl's words: "Eine axiomatisch definierte Mannigfaltigkeit kann die Eigenschaft haben, daβ jedes ihrer Objekte operativ bestimmbar ist, und zwar eindeutig. D. h. jedes Objekt, das für sie als existierend definiert ist (in die Sphäre der Existenz gehört, welche die Axiome umschreiben), ist durch die zugrunde liegenden oder eine endliche Zahl willkürlich anzunehmender bestimmter Existenzen unmittelbar oder mittelbar zu bestimmen, und zwar eindeutig. Eine solche Mannigfaltigkeit ist eine mathematische und ist definit (d.h. ihr Axiomensystem ist definit). [. . .] Relativ definit ist ein Axiomensystem, wenn es zwar für sein Existential gebiet keine Axiome mehr zuläβt, aber es zuläβt, daβ weiteres Gebiet dieselben und dann natürlich auch neue Axiome gelten. Neue Axiome, denn die bloβ alten Axiome bestimmen ja nur das alte Gebiet. Relativ definit ist die Sphäre der ganzen, der gebrochenen Zahlen, der rationalen Zahlen, ebenso der diskreten Doppelreihenzahlen (komplexen Zahlen). Absolut definit nenne ich eine Mannigfaltigkeit, wenn es keine andere Mannigfaltigkeit gibt, welche dieselben Axiome hat wie sie (alle zusammen). Kontinuierliche Zahlenreihe, kontinuierliche Doppelzahlenreihe" (Schuhmann & Schuhmann 2001, 101–102).

[33]Hilbert 1900, 1094.

[34]*Op. cit.*, 103.

[35]*Op. cit.*, 102.

[36]Husserl 1950, §72.

[37]Mahnke 1977, 80.

of isomorphism and formal deduction the exact nature of Husserl's notion of completeness is not clear. Further complications arise from the fact that the exact content of Husserl's lectures is not known either.[38, 39]

11.5 Dietrich Mahnke on Husserl and Hilbert

An interesting angle to Husserl's and Hilbert's views can be obtained from the writings of Dietrich Mahnke who had studied with both Hilbert and Husserl in Göttingen in 1902–1906. He wrote his dissertation on Leibniz under Husserl's supervision and defended it in 1922. Husserl and Mahnke had a long correspondence, to the effect that in 1921 Husserl wrote to Mahnke that he, Mahnke, was closer to him (Husserl) than any of his other students. This makes it particularly curious that Mahnke, in 1923, wrote that it is not Hilbert's early work but Hilbert's later axiomatics "in its latest exposition (first presented in lectures in Copenhagen and Hamburg 1921)" that is the most complete 'foundation' of mathematics "which comes closest to Husserl's phenomenological foundation. . . . ".[40]

Hilbert's reliance on sensuous signs in his 1922 lectures resembles Husserl's early approach to calculations in the *Philosophie der Arithmetik* (1891). Husserl however gave up the view by the time of the turn of the century. In the beginning of the century Husserl was well aware of Hilbert's views about mathematics and it would be easy to see Husserl's views of mathematics to be compared to Hilbert's axiomatics rather than to his proof-theory.[41] Thus Mahnke's claim may sound surprising, for in the lectures of 1921 Hilbert first publicly discussed what was to become his 'Program.' Indeed, Mahnke was so inspired by the

[38] Hence, it has given a rise to several competing interpretations. See for example Lohmar (1989), Da Silva (2000), Centrone (2010), Hartimo (2007), Recently Mitsuhiro Okada has defended a computational view of Husserl's completeness (Okada 2013).

[39] This paper was originally written in 2013 and represents my thoughts about the matter then. More papers on the topic have been published since, most notably, Da Silva (2015) and Hartimo and Okada (2015), and most recently in Hartimo (2016).

[40] Mahnke 1977, 77.

[41] Within a few months after Husserl's Definitheit lectures Hilbert showed Husserl his so called Memoir, the second foundations to geometry, on which Husserl took detailed notes (cf. Hartimo 2008). Husserl's interest in it, like Hilbert's, shows his unprejudiced interest in different kinds of axiomatic systems. Husserl was also well aware about the set theoretical paradoxes that plagued Hilbert's school. Zermelo's version of 'Russell's paradox' has been found written down by Husserl (Husserl 1979, 399). Hilbert also showed Husserl his correspondence with Frege about the nature of the axioms in geometry. Husserl's comment to the exchange is that Frege does not understand Hilbert's axiomatic foundations of geometry (Husserl 1970, 447–451). Husserl was also aware of the contents of Hilbert's 1905 lectures thanks to Dietrich Mahnke, who sent the lecture notes for him. In that connection Husserl expressed the wish, „recht viel aus Hilberts Darstellungen zu lernen, wie es ja eigentlich selbstverständlich ist" (Husserl Archive Leuven Mitteilungsblatt 2013, 15).

phenomenological nature of Hilbert's approach that he thought that phenomenology could be introduced by discussing Hilbert's view of formal mathematics. The paper in which he sought to do that, "Von Hilbert zu Husserl: Erste Einführung in die Phänomenologie, besonders der formalen Mathematik," appeared in 1923.

Mahnke views Hilbert's new approach as a continuation of the earlier task of axiomatics to achieve 'existence proofs' through the establishment of the consistency of systems of axioms. Mahnke describes Hilbert's proof-theoretical approach and explains how Hilbert now mounts a step higher and formalizes the statements about numbers and the deductive connections between them. He explains Hilbert's aim correctly to be to bring the "non-existence of a contradiction … available to 'inspection' in the true sense of the word."[42] Mahnke goes on[43]:

> The definitive foundation of mathematics is thus attained, according to Hilbert, by way of a 'proof critique' or 'metamathematical' theory of the mathematical process of inference, analogous to the philosophical 'critique of reason' or epistemology—I would almost say by way of a 'Copernican revolution', equivalent to the Kantian one from the object to the subject of mathematical knowledge or, in Husserl's more precise terminology by way of going back from noema to noesis, from the intended objects to the intending acts of reason.

For Mahnke, Hilbert's metamathematics is thus a shift from a focus on mathematical theories to the acts of constructing them. But, as Mahnke writes,[44]

> there always still remains for philosophers a task which lies in another dimension: that of elucidating epistemologically the inner nature and the true sense of the axiomatic method and its objects. For example, what really are those 'things' of which Hilbert speaks? Not, of course, physical realities; but neither are they ideal concepts like those of intuitive geometry; nor are they individual thought-entities (a 'particular' triangle), nor generic concepts of such individuals ('the' triangle). Rather, they are, mere conceptual skeletons without the covering of sensory material: emptied thing-forms, similar to the 'variable functions with empty places f(*)' which Hilbert introduces. They are interchangeable carriers of relations, mere supports which one can take to be of any arbitrary constitution, provided relations of the same formal-logical character can be 'fastened onto' them.

Mahnke thus holds with Husserl's view expressed in *Prolegomena* that philosopher's task is to examine the essence of mathematical theories in "another dimension". As an example of the phenomenologist's task Mahnke takes up the question of what are the things Hilbert speaks of. According to him, they are abstract objects in structures. To Mahnke, the phenomenological analysis shows that Hilbert's attempt to bring existence proofs "under inspection" is for this reason fundamentally misguided. Mahnke explains that "[t]he system of formal arithmetic is thus not really a definite scientific discipline, but only the logical form of a theory which is employed in all 'logically isomorphic' or 'formally equivalent' disciplines as their common deductive scaffolding, despite their intuitive

[42]Mahnke 1977, 78.

[43]*Op. cit.*, 79.

[44]*Loc. cit.*

incomparability."[45] Mahnke then goes on to point out that "this elucidation of Hilbert's 'things' makes it immediately clear that it is philosophically wrongly expressed when Hilbert calls the intuitive numerals the true objects of arithmetic."[46] In other words, Hilbert's formalist analysis of the objects of mathematics as constructed from sensuous signs is fundamentally misguided. However, for him, Hilbert is phenomenologically correct when he demands the givenness of 'extra-logical discrete objects' as the foundation of the arithmetically evident. According to Husserl's *Logical Investigations*, in Mahnke's view, only singular real individuals are sensorily perceived on the basis of which "higher acts of reason" are constructed:

> But on the basis of these sense perceptions higher acts of reason are constructed, such as ... colligation or counting; and it is precisely in these logical, but sensorily 'founded' experiences that aggregates and numerical totals are 'manifestly given' us, and that what is actually meant by the utterances 'and' and 'two' attains to a fully accomplished 'categorial intuition.'

Hence, Hilbert should not say that mathematics is about sensuous signs. Instead, numerals "'express' formal-logically 'isomorphically' and yet at the same time intuitively-clearly all the purely arithmetical relations, such as ordering and connection, among all arbitrary countable things."[47] Thus in phenomenology these signs have a meaning: "True objects of arithmetic are simply these logical relations."[48] In other words, according to Mahnke, Husserl in the *Logical Investigations* holds that the arithmetical theory, built upon categorial intuition of sensible objects, defines numbers, so to say, up to isomorphism. Numerals then refer to these pure numbers defined by the categorical theory. Furthermore he holds that Hilbert's problem is that he held the extra-logical sensory objects to be signs, which is a wrong philosophical analysis of the essence of numbers.

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11.6 Formal and Transcendental Logic 1929

Let us now return to Husserl's view. In the *Formal and Transcendental Logic* (1929) Husserl claims to give "definitive clarification of the sense of pure formal mathematics ... , according to the prevailing intention of mathematicians: its sense, namely, as a pure analytics of non-contradiction, in which the concept of truth remains outside the theme".[49] Presumably 'the prevailing intention,' to at least some degree, means Hilbert's intention, possibly also that of Zermelo who also was at the time in Freiburg. Husserl was still in friendly terms with Hilbert, having visited

[45] *Op. cit.*, 80.
[46] *Op. cit.*, 80–81.
[47] *Op. cit.*, 81.
[48] *Op. cit.*, 82.
[49] *FTL*, 11.

Hilbert at least once in 1928.[50] It is rather likely that on that occasion he and Hilbert discussed the views that came out in print as the *Formal and Transcendental Logic* (1929).

The "other dimension" mentioned by Mahnke above obviously amounts to transcendental phenomenological clarification. Accordingly, elucidating the inner nature and the true sense of the axiomatic method and its objects epistemologically, and for example, what really are those 'things' of which Hilbert speaks of, are roughly among the tasks for what Husserl called transcendental logic. Its task is to ask transcendental questions about logic, the task that Kant, according to Husserl, failed to address. Such reflection examines logicians' intentions, their aimings and fulfillment – "the activity that is hidden … throughout the naive doing and only now becomes a theme in its own right – we examine that activity after the fact".[51] Husserl's motivation is very similar to the way in which Hilbert characterizes his motivation to formulate his proof theory: "The fundamental idea of my proof theory is none other than to describe the activity of our understanding, to make a protocol of the rules according to which our thinking actually proceeds."[52] However, Husserl wants to examine the mathematicians' implicit, hidden presuppositions transcendentally, not proof-theoretically. According to Husserl, the transcendental logic helps to clarify the "internal shiftings of intentionality" that lead to equivocations.

Such examination has, for example, led Husserl to distinguish between three strata of logic and the corresponding three different evidences with their corresponding modes of empty expectation and fulfillment.[53] With such clarification, Husserl claims, the fundamental concepts of the sciences are clarified and criticized so that there should be no paradoxes.[54] The three levels of logic are grammar, formal mathematics (logic of non-contradiction) and applied mathematics (truth-logic). The first level defines the grammar of pure logic, what he also calls a theory of the pure forms of significations. It is a theory of how from certain fundamental forms a system of all conceivable judgment-forms emerges.[55] The second level is the logic of non-contradiction. Theories of this level are mathematical theories that are non-contrary. There is no need for them to be applicable to the world. The third level, the logic of truth, is the level that requires empirical applicability.[56]

Husserl exemplifies the different layers by means of Euclidean geometry that can be regarded either purely mathematically or else as something that aims at true

[50]During his Freiburg years Husserl considered himself as an old friend of Hilbert's household. Husserl visited Hilbert about which he reported to Heidegger that Hilbert's reception had been very friendly, „Sehr freundschaftlich kam uns Hilbert entgegen" (Husserl to Heidegger 9.5.1928, Husserl-Archive Mitteilungsblatt 26, 2013, 16).

[51]*FTL*, §69.

[52]Hilbert 1927, 475.

[53]*FTL*, §70a.

[54]*Op. cit.*, §71.

[55]*Op. cit.*, §13.

[56]For a nice exposition of Husserl's conception of logic in *FTL*, see Cavaillès 1970, 386–409.

judgments about the world (even though still certain idealizations are required). In the latter case, in the logic of truth, "one must draw fullness of adequation, not from evidence of the judgment-senses, but instead from evidence of the 'matters' or 'affairs' corresponding to them," Husserl writes.[57] In addition, much later in the text, Husserl also defines a supplement by means of which the complex judgments will be reduced to judgments about individuals. Such a supplement is located between the logic of non-contradiction and the logic of truth. I will next discuss in more detail Husserl's logic of non-contradiction and Husserl's reduction. My intention is to show that the former, the logic of non-contradiction consists of theories with a structure, whereas the syntactic reduction is analogous to Hilbert's proof-theory in that it shows how the former is accessible from our concrete meaningful experiences. I will not discuss Husserl's logic of truth in this connection. It would however offer another interesting connection between Husserl and Hilbert, for it shows how for Husserl mathematics is applied to the empirical world.

11.7 The Logic of Non-contradiction: Formal Mathematics

Like in Hilbert's axiomatics, in the mathematics of non-contradiction as Husserl conceives it, mathematical 'existence' derives from consistency. Husserl holds that the mathematician does not need to think that the multiplicities for example, exist concretely.[58] For him, for a "*'pure' formal mathematics, there can be no cognitional considerations other than those of 'non-contradiction',* of immediate or mediate *analytic consequence or inconsistency*, which manifestly include all questions of mathematical *'existence'* ".[59]

The evidence related to the formal mathematics is called "distinctness" [Deutlichkeit]. The distinct judgements may or may not be true of the world. In the formal mathematics the fundamental question is not whether the judgment is true or not, but: "When, and in what relations, are any judgments – as judgments, and so far as mere form is concerned – possible within the unity of one judgment."[60] Later, as a result of a more detailed analysis, Husserl explains that the distinct judgment is a proper judgment in a sense that it has a unitary, non-contradictory sense (in addition to the purely grammatical sensefulness). The source of this unity is in the ideal "existence" of the judgment-content.[61] It is rooted not only in syntactical forms but also syntactical stuffs [Stoffe].[62] At this point Husserl remarks that this is "easily overlooked by the formal logician, with his interest directed one-sidedly to the

[57] *Op. cit.*, §82.
[58] *Op. cit.*, §51.
[59] *Op. cit.*, §52.
[60] *Op. cit.*, §18.
[61] *Op. cit.*, §89a.
[62] *Op. cit.*, §89a.

syntactical—the manifold forms of which are all that enters into logical theory—and with his algebraizing of the cores as theoretical irrelevancies, as empty somethings that need only be kept identical."[63]

Husserl thus wants to analyze the contentful mathematics as something that contains syntactical stuff in addition to pure form. The above quote appears to be directed against Hilbert, who in Husserl's view seems to be one-sided in his focus on the syntactical side of mathematics alone. In other words, for Husserl formal logic is not mere syntax, but the content of the judgments should be taken into account as well.

Husserl's discussion of syntactical forms and stuffs is notoriously difficult to understand. One way of trying to make sense out of them is to understand them and thus the unity of judgment-content proto-model theoretically: According to Husserl, all judging presupposes a harmonious unity of possible experience. In this harmony, everything has 'to do' materially with everything else. Husserl writes, "in respect of its content, every original judging and every judging that proceeds coherently, has coherence by virtue of the coherence of the matters in the synthetic unity of the experience, which is the basis on which the judging stands".[64] These matters in the synthetic unity of the experience are presumably the syntactical stuffs. The synthetic unity of experience is further something one experiences when provided with a structure, and the evidence related to this experience is then evidence of distinctness.

In Husserl's words, then a consistent theory does not merely have a grammatical unity, but "*coherence by virtue of the coherence of the matters in the synthetic unity of experience*," which Husserl also calls a universe of possible experience [ein Universum möglicher Erfahrung].[65] Its "cores are congruous in respect of sense—that is: all judgments that fulfill the conditions for unitary sensefulness."[66] Interestingly, Husserl adds that

> [w]e do not intend to say in advance that there can be only one universe of possible experience as the basis for judgment, and that therefore every intuitive judgment has the same basis and all judgments belong to a single materially coherent whole. To reach a decision about that would require a separate investigation.[67]

Husserl thus leaves open the possibility of several "universes of possible experiences."

The evidence obtained from the structure is evidence of distinctness. It has the same expectation and fulfillment structure as the other evidences.[68] Also the law of the excluded middle holds of the distinct judgments. This means that they can be brought to either adequation, in which case the judgment explicates

[63] *Op. cit.*, §89b.

[64] *Loc. cit.*

[65] *FTL*, §89 a–b.

[66] *Op. cit.*, §90.

[67] *Op. cit.*, §89b.

[68] *Op. cit.,* §70a.

and apprehends categorially what is given in harmonious experience, or to the negative of adequation, in which case what it predicates conflicts with something experienced.

These universes of possible experiences, or harmonious unities of possible experience (or a unitary material province, ein einheitliches sachliches Gebiet, as he also calls it in §92) function like models in the contemporary model theoretic semantics. Husserl's harmonious unity of possible experience thus realizes or satisfies the non-contradictory theory. The experience of distinctness derives from the existence of a model in which the judgment can be brought to adequation, i.e., "satisfied" by means of categorial intuition.[69]

Husserl's conception of formal mathematics is thus in agreement with Hilbert's view in so far as the latter viewed mathematics as a study of structures. Husserl's view seems also close to Zermelo's view. Zermelo held that the models, what he called 'substrates,' are presupposed in mathematics.[70] But while Hilbert wanted to analyze the inferences syntactically with his proof theory, Husserl held that the formal content of mathematics cannot be analyzed away and that the models are presupposed in mathematics. In the *Formal and Transcendental Logic* (§33) Husserl also discusses a possibility of building a

> discipline comprising deductive games with symbols, which does not become an actual theory of multiplicities itself, one builds only a discipline comprising deductive games with symbols, which does not become an actual theory of multiplicities until one regards the game-symbols as signs for actual Objects of thinking—units, sets, multiplicities—and bestows on the rules of the game the significance of law-forms applying to these multiplicities.

This is easy to read as a critique of Hilbert. Husserl, like Mahnke before, holds that the numerals should not be regarded as objects of mathematics. Husserl explicitly points out that

> we must not define merely in terms of signs and calculational operations—for example: 'It shall be allowed to manipulate the given signs in such a manner that the sign b + a can always be substituted for a +b'. Rather we must say: 'There shall obtain among the *objects* belonging to the multiplicity (conceived at first as only empty Somethings, 'Objects of thinking') a certain *combination-form* with the law-form a+b=b+a'—where equality has precisely the sense of actual *equality* such as belongs to the categorial logical forms.[71]

Nevertheless, using symbols as a technique facilitating the calculations is not a problem for Husserl. But whereas for Hilbert such technique expresses our thinking

[69] 'Categorial intuition' is Husserl's term for perception of formal structures, typically states of affairs. From Husserl's remarks it is difficult to say what everything could be an object of categorial intuition. Dietrich Mahnke and Oskar Becker discussed this matter and disagreed about it: Becker held that categorial intuition is restricted to human consciousness and that we cannot intuit transfinite elements. Mahnke thought that this is not the case, and that the consciousness in question is an ideal consciousness (Mancosu & Ryckman 2010, 350–355). Given Husserl's overall non-revisionist attitude Mahnke's view about the matter seems to be closer to Husserl's intentions.

[70] Ebbinghaus 2007, 156.

[71] *FTL*, §34.

irreducibly that is not the case for Husserl. Thus when Hilbert explains that "[t]his formula game is carried out according to certain definite rules, in which the technique of our thinking is expressed," Husserl would agree. But when Hilbert continues that "[t]he fundamental idea of my proof theory is none other than to describe the activity of our understanding, to make a protocol of the rules according to which our thinking actually proceeds,"[72] Husserl would point out that when so doing, Hilbert ignores the formal content of mathematics and the evidence of *Deutlichkeit* related to it. Transcendental examination however makes it explicit that such formal content cannot be ignored. Without the transcendental clarification of the evidences, there will be equivocations "which we cannot remove while confining out attention to the language itself and simply examining it with respect to the significations to which it points associationally."[73] Husserl thus seems to suggest that Hilbert's way of founding mathematics with concretely intuitable signs will not solve the equivocations and paradoxes that have their sources in the contents of the judgments, not only in their syntactic formulation.

11.8 Syntactic Reduction

Despite of the above criticism Husserl seems to agree with Hilbert that something like Hilbert's proof-theoretical reduction is needed. Indeed, in the *Formal and Transcendental Logic*, too, something similar can be found. Husserl establishes a "reduction" of formal mathematics to logic of truth and then to judgments about individuals. According to Husserl, "*any actual or possible judgment leads back to ultimate cores* when we follow up its syntaxes; accordingly that it is a syntactical structure built ultimately, though perhaps far from immediately, out of *elementary cores, which no longer contain any syntaxes*".[74] By means of this kind of reduction the ultimate substrates, i.e., absolute subjects, ultimate predicates, ultimate universalities, and ultimate relations, are reached. According to Husserl, these are of no interest for general mathematics, but if we are interested in truth, then this is important: "because ultimate substrate-objects are *individuals*, about which very much can be said in formal truth, and *back to which all truth ultimately relates*".[75] Similar reduction takes place among truths, so that every true judgment ultimately relates to individual objects. These individual objects are then objects of our experiences about which we can predicate something as in "The man is pale", "The paper is white", etc. The constitution of such objectivities and the further structures related to such experiences is then the topic for phenomenological

[72]Hilbert, 1927, 475.

[73]*FTL*, §70.

[74]*FTL*, §82.

[75]*FTL*, §82.

analyses. That is what Husserl in the above quote refers to as formal truth as he gives the "time-form" as an example.

Husserl thus agrees with Hilbert that mathematics presupposes intuition of extra-mathematical objects, and hence a reduction to primitive judgments about individual objects, although not numerals, is needed. This takes place by means of a *transitional link* between mathematics of non-contradiction, i.e., formal mathematics, and truth-logic. It is an analytic reduction of mathematics to ultimate substrate-objects that are individuals.

What is curious is that Husserl claims that for formal mathematics such a reduction is of no particular interest.[76] For him mathematics as such is not in need of a direct proof-theoretical consistency proof, but that the above described model theoretical approach is quite sufficient for the formal mathematicians. However, he claims that the reduction to the ultimate cores is important for the question of truth. In the logic of truth the judgments are viewed as striving for truth, anticipating the possible fulfillment by means of intuition. The focus of judging is in the states of affairs about which something is judged.[77] The logic of truth is thus about the world, and thus the reduction to judgments about individuals show how mathematics is related to our judgments about the world. The motivation for it is primarily philosophical: to understand the nature and essence of mathematical theories so that it is all as if "available to inspection", as for Hilbert. Such reduction thus starts to resemble in its motivation very much what Wilfried Sieg has termed Hilbert's 'structural reduction,' in which the abstract structures are projected into the finitist domain, "ever more encompassing part of mathematics to a fixed, elementary, and meaningful part of itself".[78] Such a reduction thus shows how the abstract mathematical structures are accessed from a concrete meaningful experience.

11.9 Conclusion

Husserl shares Hilbert's approach to mathematics as "mathematics first," approach to formal mathematics that deals with axiomatically characterized abstract structures. Thus in accordance to the division of labor between mathematicians and philosophers, Husserl left the task of constructing the theories to Hilbert and his colleagues, while Husserl saw as his task to examine their essence and their presuppositions. Husserl developed the view further in his *Formal and Transcendental Logic,* which appropriates many of Hilbert's insights but is also critical about his view. In particular, Hilbert should not conceive his reduction purely syntactically and base it on sensuous signs. Rather mathematics is about structural objects and thus the reduction should take into account the "formal content" of the judgments

[76] *FTL*, §82.

[77] *FTL*, §19.

[78] Sieg 2013, 17.

as well. Husserl seems to think that for mathematics model theoretical consistency proofs are entirely adequate. Yet Husserl offers us an analogous syntactic-semantic reduction to objects as a transitional link between formal mathematics and truth-logic. This however is not Husserl's view of how the direct consistency proof should be established, but rather a philosophically motivated analysis of how mathematical theories relate to our experiences.

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